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THE LATTICE OF FULLY INVARIANT SUBGROUPS OF A REDUCED COTORSION GROUP

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Abstract. The present work considers the lattice of fully invariant subgroups of a reduced cotorsion group, when torsion part is a countable direct sum of torsion-complete groups. It is shown that the lattice is isomorphic to the lattice of filters of a semi-lattice constructed of infinite matrices and indicators.

რეზიუმე. განიხილება რედუცირებული კოგრეხვითი ჯგუფის სავსებით ინვარიანტულ ქვეჯგუფთა მესერი, როცა გრეხვითი ნაწილი გრეხვითად სრული ჯგუფების თვლადი პირდაპირი ჯამია. ნაჩვენებია, რომ ეს მესერი იზომორფულია უსასრულო მატრიცებისა და ინდიკატორებისაგან აგებული ნახევარმესრის ფილტრთა მესრისა.

In the present work we consider the problems of the theory of abelian groups. Throughout the paper, under the word "group" we will mean the additively written abelian group. The use will be made of the notation and terminology from monographs [2] and [3].

By p we denote a fixed prime number. Z and Q are, respectively, additive groups of integer and rational numbers. Q_p^* denotes the ring of p-adic integers, J_p is its additive group. A *mixed group* involves both the nonzero elements of finite order and the elements of infinite order. A subgroup A of the group B is said to be *fully invariant* if B is mapped into itself for any endomorphism of the group A. Such are, for example, the subgroups $nA = \{na \mid a \in A\}$, $A[n] = \{a \in A \mid na = 0\}$, $n > 0$, $n \in Z$ and the torsion part of a group A.

A group D is said to be *divisible* (respectively *p-divisible*), if the equality $nx = a$ (respectively $p^n x = a$) has a solution in D for any natural number n and any $a \in D$. Such are, for example, the groups Q, Q/Z, the *quasicyclic group* $Z(p^\infty)$, i. e. the group generated by the elements $c_1, c_2, \dots, c_n, \dots$, where $pc_1 = 0$, $pc_2 = c_1, \dots, pc_{n+1} = c_n, \dots$. Every divisible group is a direct sum of isomorphic copies of Q and $Z(p^\infty)$ (for various p), and since every abelian group expands into a direct sum of a divisible and a *reduced group* (i. e. a group containing no nonzero divisible subgroups), the problem

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of description of various abelian groups comes to the case of describing the corresponding reduced groups.

A subgroup G of a group A is said to be *pure* (respectively, *p-pure*) if every equation $nx = g \in G$ (respectively, $p^n x = g$) which has a solution in the whole group A has a solution in G as well. For example, every direct summand of a group is a pure subgroup.

A subgroup B of the group A is said to be a *p-basic subgroup*, if the following three conditions are fulfilled: 1. the subgroup B is a direct sum of cyclic p-groups and infinite cyclic groups; 2. B is a p-pure subgroup of the group A; 3. the quotient group A/B is a p-divisible group.

A group A is said to be *algebraically compact* if it splits out as a direct summand from every group G containing it as a pure subgroup. A group A is complete in the Z-adic topology (i. e. in the topology in which the subgroups nA, $n \in Z$, $n \neq 0$ form the base of zero neighborhoods) if and only if it is a reduced algebraically compact group (see [2], §39).

Torsion-complete p-groups are defined as torsion parts $t(\widehat{B}) = \overline{B}$ of p-adic completions $\widehat{B}$ of direct sums B of cyclic p-groups. (In the p-adic topology the subgroups $p^n A$, $n \in Z$, $n \neq 0$ form the base of zero neighborhoods).

If $a \in A$, then the largest nonnegative integer r for which the equality $p^r x = a$ has a solution $x \in A$ is called the *p-height* $h_p(a)$ of the element a. If the equality $p^r x = a$ has a solution for any r, then a is called an element of infinite p-height, $h_p(a) = \infty$. The zero is of infinite height with respect to any prime number. A p-group A is said to be *separable* if it contains no elements of infinite height, i. e. the *Ulm subgroup* $A^1 = \bigcap\limits_{n=1}^{\infty} p^n A = 0$.

The investigation of the lattice of fully invariant subgroups of a group is an important task of the theory of abelian groups. For sufficiently wide class of p-groups, this problem has been studied by R. Baer [1], I. Kaplansky [6], P. Linton [9], R. Pierce [15], D. Moore and E. Hewett [13], etc. The works due to A. Mader [11], R. Göbel [4], A. Moskalenko [14], S. Grinshpon and Krylov [5], Misyakov [12], and other authorsvare dedicated to the study of this topic in torsion free and mixed groups.

The lattices of fully invariant subgroups in the class of cotorsion groups are scarcely studied. A group A is said to be a *cotorsion group* if its extension by means of any torsion free group C splits: $\mathrm{Ext}(C, A) = 0$. Importance of the class of cotorsion groups in the theory of abelian groups can be explained by two facts: for any groups A, B the group $\mathrm{Ext}(A, B)$ is a cotorsion group; every reduced group A is isomorphically embedded into the group $A^{\bullet} = \mathrm{Ext}(Q/Z, A)$, the so-called *cotorsion hull* of the group A, and in addition, $A^{\bullet}/A$ is a divisible torsion free group. Every reduced cotorsion group A decomposes into a direct sum $A = T^{\bullet} \oplus C$, where $T^{\bullet} \cong \mathrm{Ext}(Q/Z, T)$, $T = tA$ is the torsion part of the group A, and $C \cong \mathrm{Ext}(Q/Z,\, A/Z)$ is an algebraically compact torsion free group. If $T = \underset{p}{\oplus} T_p$ is the decomposition into the direct sum of primary components, then $\mathrm{Ext}(Q/Z, T) \cong \underset{p}{\Pi}\mathrm{Ext}(Z(p^\infty), T_p) = \underset{p}{\Pi} T^{\boldsymbol{\cdot}}_p$, $\mathrm{Ext}(Q/Z,\, A/T) \cong \underset{p}{\Pi}\mathrm{Ext}(Z(p^\infty), A/T) = \underset{p}{\Pi} C_p$ and

$$A = \underset{p}{\Pi}(T^{\boldsymbol{\cdot}}_p \oplus C_p). \tag{1}$$

Thus the study of cotorsion groups reduces in a considerable extent to that of groups of the type

$$A_p = T^{\boldsymbol{\cdot}}_p \oplus C_p, \tag{2}$$

where T_p is a p-group and C_p is an algebraically compact torsion free group. As is shown in [2] (§§40, 54), $T^{\boldsymbol{\cdot}}_p$, C_p and, hence, A_p are *p-adic modules*, i. e. modules over the ring Q^*_p.

The study of fully invariant subgroups in cotorsion groups is still more important because endomorphisms in this class of groups are defined by their effects on the torsion part and, as is shown in [10, Theorem 3.3], in the mixed groups, the rings of endomorhisms $E(A) \cong E(tA)$ if and only if A is a fully invariant subgroup of the cotorsion hull $(tA)^{\boldsymbol{\cdot}}$.

In investigations of lattices of fully invariant subgroups of a group, the notions of the indicator and of a fully transitive group are of essential use.

The *p-indicator* of an element a of a group A is the increasing sequence of ordinal numbers

$$H_A(a) \equiv H(a) = \big(h(a), h(pa), \dots, h(p^n a), \dots\big),$$

where h denotes the *generalized p-height* of the element a, i. e. $h(a) = \sigma$ if $a \in p^\sigma A \backslash p^{\sigma+1} A$ and $h(0) = \infty$ (of course, if $h(p^n a) = h(0) = \infty$, then $h(p^{n+1} a) = \infty$). On the set of indicators we can introduce the ordering

$$H(a) \le H(b) \Longleftrightarrow h(p^i a) \le h(p^i b), \quad i = 0, 1, 2, \dots$$

A reduced p-group is said to be *fully transitive* if for its arbitrary elements a and b, for $H(a) \le H(b)$, there exists an endomorphism φ of the group such that $\varphi a = b$. In fully transitive groups, using the aforementioned indicators, we study the lattice of fully invariant subgroups. In particular, I. Kaplansky has shown that every such subgroup of a group A has the form

$$A(u) = \big\{a \in A \,|\, H(a) \ge u\big\},$$

where $u = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ is an increasing sequence of ordinal numbers and symbols ∞, satisfying the condition: if between σ_n and σ_{n+1} there is a jump (i. e. $\sigma_{n+1} > \sigma_n + 1$), then A contains an element of order p and height σ_n (see [3, Theorem 67.1]).

A. Mader [11] has shown that an algebraically compact group is fully transitive, and using the aforementioned indicators, he described the lattice of fully invariant subgroups of an algebraically compact group. In addition, in a generalized form he pointed out the conditions whose fulfilment enables one to describe the lattice of fully invariant submodules.

Theorem 1 (A. Mader). *Let A be a module over a commutative ring R, Δ the lattice of all its fully invariant submodules, Ω some meet-semilattice and $\Phi : A \to \Omega$ a mapping possessing the following properties:*

1) Φ is surjective;

2) $\Phi(fa) \geq \Phi(a)$, $\forall\, a \in A$ and $f \in EndA$;

3) $\Phi(a+b) \geq \Phi(a) \wedge \Phi(b)$;

4) if $\Phi(a) \geq \Phi(b)$, then there exists the endomorphism f of the module A such that $f(b) = a$;

5) if $C \in \Delta$, then for any $a, b \in C$ there exists $c \in C$ such that $\Phi(c) = \Phi(a) \wedge \Phi(b)$.

Then the set Ω^* of all filters of Ω, ordered with respect to the inclusion, is a lattice, and the mapping $\alpha : \Omega^* \to \Delta$ defined by the rule $\alpha(D) = \{a \in A \,|\, \Phi(a) \in D\}$ is a lattice isomorphism.

Just as in p-groups, we define the notion of full transitivity in the group $T^{\bullet} = \mathrm{Ext}(Z(p^\infty), T)$. If T is a torsion-complete group, then its cotorsion hull is an algebraically compact group (see [2, §56]) and, as mentioned above, is fully transitive.

A. I. Moskalenko [14] has proved that if T is a direct sum of cyclic p-groups, then $T^{\bullet}$ is likewise fully transitive, and all the conditions of Theorem 1 are fulfilled. Hence in this case the lattice Ω^* of indicator filters describes the lattice of fully invariant subgroups. A natural generalization of torsion-complete groups and of direct sums of cyclic p-groups are direct sums of torsion-complete groups. As the author has shown [7], in this class of groups, if the sum is infinite, the cotorsion hull is not fully transitive. Therefore owing to the condition 4 from Mader's theorem, the indicators cannot be used to describe the lattice of fully invariant subgroups.

In [8], the lattice of fully invariant subgroups of the group $T^{\bullet}$ is studied in the case where T is a countable direct sum of torsion-complete p-groups:

$$T = \bigoplus_{j=1}^{\infty} \overline{B}_j, \tag{3}$$

where B_j is the basic subgroup of $\overline{B}_j$, and $B = \bigoplus_{j=1}^{\infty} B_j$ is the basic subgroup of T.

For a separable p-group T, the elements of the cotorsion hull $T^{\bullet}$ were represented by A. I. Moskalenko [14] in the form of countable sequences

$$T^{\bullet} = \big\{(a_0, a_1 + T, \dots, a_i + T, \dots) \,|\, a_i \in \widehat{T},\, pa_{i+1} - a_i \in T,\, i = 0, 1, \dots\big\}.$$

For such writing of elements, one can easily calculate the height and the indicator. In particular, if $a = (a_0, a_1 + T, \dots)$, then

$$H_{T^{\bullet}}(a) = \begin{cases} H_{\widehat{T}}(a), \ \text{if order } \mathcal{O}(a_0) = \infty; \\ (h_{\widehat{T}}(a_0), h_{\widehat{T}}(pa_0), \dots, h_{\widehat{T}}(p^{n-1}a_0), \omega+m, \omega+m+1, \dots); \\ \text{if } a_0 \in \widehat{T}\backslash T, \ \mathcal{O}(a_0) = p^n, \ \mathcal{O}(a_0 + T) = p^{n-m}; \\ (h_T(a_0), h_T(pa_0), \dots, h_T(p^{n-1}a_0), \\ \qquad\qquad \omega + n + k, \omega + n + k + 1, \dots); \\ \text{if } \mathcal{O}(a_0) = p^n, \ a_0, a_1, \dots, a_k \in T, \ a_{k+1} \notin T; \\ H_T(a_0), \ \text{if } a_i \in T \ \text{for any } i, \end{cases} \tag{4}$$

where ω is the least infinite ordinal number.

The torsion part of the group $T^{\bullet}$ consists of sequences $(c, T, T, \dots)$, where $c \in T$, and the first Ulm subgroup involves sequences of type $(o, a_1 + T, \dots)$. Let $B = \bigoplus_{\alpha\in J} < x_\alpha >$ be a fixed basic subgroup of the separable p-group T. If $a \in T^{\bullet}$, $a = (a_0, a_1 + T_1, \dots)$, then in the group B there exists a sequence of elements (b_i), $i = 0, 1, \dots$ such that for any i,

$$b_i = \sum_{j=1}^{s} m_j x_{\alpha j}, 0 \le m_j < p \quad \text{and} \quad a_i = \lim_{n\to\infty} \Big(\sum_{s=0}^{n} p^s b_{i+s}\Big). \tag{5}$$

Such representation of the element a is called *canonical*. One says that the sequence (b_i) corresponds to the canonical representation of the element a. The following statements are valid (see [14]).

Proposition 1. *If $H_{T^{\bullet}}(a) = (k_0, k_1, \dots)$, (b_i) is a sequence corresponding to the canonical representation of an element a, and between k_i and k_{i+1} there is a jump, then in the decomposition of $b_{k_i - i}$ with respect to the basis $\{x_\alpha | \alpha \in J\}$ there is x_α of order p^{k_i+1}.*

Proposition 2. *If $H_{T^{\bullet}}(a)$ is a sequence of natural numbers, then it has an infinite number of jumps.*

Let the group T be of the form (3) and $a \in T^{\bullet}$ (see (5)). By π_i we denote projection of the group T onto the direct sum $\overline{B}_i$, and consider the sequence

$$\pi_i(b_j) = (b_{ij}), \quad j = 0, 1, 2, \dots.$$

For every $i \geq 1$, the sequence $b_{i0}, b_{i1}, \dots$, for fixed j defines the element $a_{ij} = \lim_{n\to\infty} \sum_{s=0}^{n} p^s b_{ij+s}$, whereas the elements $a_{i0}, a_{i1}, \dots$ of the group $\widehat{B}_i$ define the element $a^{(i)} = (a_{i0}, a_{i1} + T, a_{i2} + T, \dots)$ of the group $T^{\bullet}$. Obviously,

$$a_j = \lim_{n\to\infty} \sum_{i=1}^{n} a_{ij}.$$

Note that using the given element a from (5), the elements $a^{(i)}$, $i = 1, 2, \dots$ are defined uniquely, since if $(b'_j)_{j\geq 0}$ is another sequence corresponding to the element a, and $(a_0, a_1 + T, \dots) = (a'_0, a'_1 + T, \dots)$, then $a_0 = a'_0$ (see [14, §§1,2]), and if $a_k - a'_k \in T$, then $\widehat{\pi}_i(a_k) = a_{ik}$, $\widehat{\pi}_i(a'_k) = a'_{ik}$ and $a_{ik} - a'_{ik} = \widehat{\pi}_i(a_k - a'_k) \in T$, $k = 1, 2, \dots$, where $\widehat{\pi}_i$ denotes the induced projection on the group $\widehat{B}_i$.

To every element $a \in T^{\cdot}$ we assign the matrix

$$\Phi(a) = \|H(a_{i0})\|_{i\geq 0},$$

where $H(a_{00}) = H_{T^{\cdot}}(a)$, and $H(a_{i0}) = H_{\widehat{T}}(a_{i0})$ for $i \geq 1$.

Definition 1. The matrix $\|k_{ij}\|_{i,j\geq 0}$ composed of ordinal numbers and symbols ∞ we call *admissible with respect to the group* T if the following conditions are fulfilled:

1. The first row is an increasing sequence of ordinal, less than $\omega + \omega$, numbers and symbols ∞, and if $k_{0j} \geq \omega$, then $k_{0n+1} = k_{0n} + 1$ for any $n \geq j$; other rows are increasing sequences of nonnegative integers or symbols ∞ (it is assumed that $\infty + 1 = \infty$).

2. If in the first row $k_{0n} = \omega + m$ is the first infinite ordinal number and $m < n$, then in an infinite number of rows there appear nonnegative integers, and there exists a row i_0 such that $k_{in-m} = \infty$ for $i \geq i_0$. When $k_{0n} = \omega + m$, $m \geq n$ then starting from some i_0, all rows are composed only of the symbols ∞.

3. If all elements in a row are nonnegative integers, then we have infinitely many jumps.

4. If between k_{ij} and k_{ij+1} there is a jump, then in the group B_i there exists a basis element of order $p^{k_{ij}+1}$ (assuming $B_0 = B$).

5. In every column $k_{ij} \to \infty$ as $i \to \infty$, and if $k_{0j} \neq \omega + m$, then $k_{0j} = \min\{k_{1j}, k_{2j}, \dots\}$, while if $k_{0j} = \omega + m$, then $k_{1j} = k_{2j} = \dots = \infty$.

Taking into account equality (3) and Propositions 1 and 2, we can see that for any $a \in T^{\cdot}$ the matrix $\Phi(a)$ satisfies the above conditions.

It follows from Definition 1 that we are concerned with three types of matrices:

$$\text{I.}\ \begin{bmatrix} k_{00} & k_{01} & \cdots \\ k_{10} & k_{11} & \cdots \\ \multicolumn{3}{c}{\dotfill} \end{bmatrix},$$

where k_{ij} are the nonnegative integers or the symbols ∞;

$$\text{II.}\ \begin{bmatrix} k_{00} & k_{01} & \cdots & \cdots & \cdots & \cdots & k_{0n-1} & \omega+m & \omega+m+1 & \cdots \\ k_{10} & k_{11} & \cdots & \cdots & \cdots & \cdots & k_{1n-1} & \infty & \infty & \cdots \\ \multicolumn{10}{c}{\dotfill} \\ k_{t0} & k_{t1} & \cdots & k_{tn-m-1} & \infty & \cdots & \infty & \infty & \infty & \cdots \\ \multicolumn{10}{c}{\dotfill} \end{bmatrix},$$

where $m < n$ and k_{ij} are the nonnegative integers (see the first sentence of point 2 in Definition 1);

$$\text{III.}\quad \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0n-1} & \omega+n+k & \omega+n+k+1 & \cdots \\ k_{10} & k_{11} & \cdots & k_{1n-1} & \infty & \infty & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{i-10} & k_{i-11} & \cdots & k_{i-1n-1} & \infty & \infty & \cdots \\ \infty & \infty & \cdots & \infty & \infty & \infty & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

where k_{ij} are nonnegative integers (see the second sentence of point 2 in Definition 1).

By Ω we denote the set of matrices admissible with respect to T and define on the set Ω a reflexive and transitive relation $\leq$ (see Definition 2).

Let $K = \|k_{ij}\|_{i,j=0,1,\dots}$ be an admissible matrix. We perform the following partitioning of rows $i = 1, 2, \dots$ of the matrix K.

Let (k_{0i_s}, k_{0i_s+1}), $s = 1, 2, \dots$ be all jumps in the row $k_{00}.k_{01}, \dots$. All j-th rows, where $k_{ji_s} = k_{0i_s}$ for some $s = 1, 2, \dots$, are referred to be of the first class. By taking minimum in every column, the remaining rows form the sequence $k_1, k_2, \dots$. If this sequence does not possess the symbol ∞, then taking into account points 1,3 and 5 of Definition 1, we find that it has infinitely many jumps which we denote by (k_{i_s}, k_{i_s+1}), $s = 1, 2, \dots$. In the second class, we combine all j-th rows, where $k_{ji_s} = k_{i_s}$, for some $s = 1, 2, \dots$, and so on. Thus we obtain the uniquely defined partitioning of rows of the matrix K into nonintersecting classes and call it *the basic partitioning of the matrix*, and the classes of partitioning we call *closed classes. The row of the class* we call a sequence which is obtained by taking the minimum in every column of rows of the given class. We say that *one class is less than or equal* to the other class if for their rows (k_i) and (k'_i) the condition $k_i \leq k'_i$, $i = 0, 1, \dots$ is fulfilled. If we take into account that owing to point 5 of Definition 1, in every j-th column $b_{ij} \to \infty$, as $i \to \infty$, it is not difficult to see that for the rows of closed classes of the basic partitioning of the matrix the condition $k_i \leq k'_i$, $i = 0, 1, 2, \dots$ is fulfilled, i.e., these *classes are linearly ordered.*

Below we present an example of an admissible matrix K. It is shown that 26 rows and the first closed class form the rows with numbers 1,2,4; the second class forms rows with numbers 3,5,6,7; the third class form rows with numbers 8,10,11; the fourth class forms rows with numbers 17,19,22,23,24,25, 26. All this is marked by Roman numerals on the right-hand side of the matrix along the corresponding rows (we mean that in places with dots there are positive integers preserving closure of visible classes).

$K =$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	...		
0	1	3	4	5	6	7	9	10	11	13	14	16	17	19	20	22	24	26	29	30	32	33	35	39	40	43	46	47	49	...	0	
0	1	3	4	5	6	7	9	11	12	13	14	16	18	19	20	23	24	26	29	30	32	33	36	40	41	44	46	47	50	...	1	I
1	2	3	4	5	6	7	10	12	13	14	15	16	17	20	21	22	25	27	29	31	32	34	35	39	40	43	46	48	49	...	2	I
1	2	3	5	7	8	9	12	13	14	16	17	19	20	21	22	24	26	28	30	31	32	34	36	39	41	44	47	48	52	...	3	II
2	3	4	5	6	7	8	9	10	11	13	14	17	18	19	23	24	25	26	30	32	33	34	35	39	40	44	47	48	51	...	4	I
2	3	4	6	7	8	9	11	12	13	15	17	20	21	22	23	24	26	27	29	31	33	35	36	40	42	47	48	49	50	...	5	II
3	4	5	6	8	9	10	11	12	14	16	18	19	20	21	22	23	25	27	29	32	33	36	37	40	41	45	48	49	51	...	6	II
3	5	6	7	8	9	10	11	12	13	15	16	18	19	20	21	23	27	28	30	33	34	35	38	40	43	44	47	50	51	...	7	II
1	3	5	7	8	9	12	13	14	15	16	18	20	22	23	25	27	29	31	33	34	35	36	39	41	43	46	57	58	59	...	8	III
2	3	5	7	8	10	12	13	15	18	19	21	22	24	25	29	30	31	33	38	39	40	43	44	46	47	49	50	52	54	...	9	IV
1	2	5	6	7	8	12	13	14	15	16	18	19	23	25	26	27	28	29	34	36	37	38	39	40	42	46	58	60	62	...	10	III
2	3	4	6	7	8	13	14	15	16	17	19	21	22	23	24	27	29	31	34	37	38	39	40	42	44	45	50	51	52	...	11	III
2	3	6	7	8	9	13	14	15	18	19	20	21	25	26	29	31	32	33	37	40	41	44	45	46	48	49	51	53	55	...	12	IV
2	6	7	8	10	16	17	18	24	25	26	27	29	30	32	33	35	37	38	40	42	43	48	53	54	55	60	61	63	65	...	13	V
3	4	5	6	7	9	12	13	16	18	20	21	22	24	26	29	31	33	34	37	38	39	43	46	47	48	50	52	53	54	...	14	IV
4	5	6	7	8	10	13	17	18	19	20	22	23	25	26	30	32	34	35	38	40	41	43	44	45	46	47	50	52	53	...	15	IV
2	6	7	8	12	15	17	19	23	24	25	26	28	30	33	34	35	36	38	41	43	44	49	52	53	56	60	61	63	64	...	16	V
5	7	10	11	16	17	18	19	24	25	27	28	36	37	39	40	43	46	47	49	51	53	58	59	60	62	65	66	67	68	...	17	VI
3	6	7	8	11	15	16	17	23	25	26	27	29	31	32	34	35	36	37	40	42	45	50	52	54	56	61	62	64	66	...	18	V
5	7	10	11	16	17	18	19	24	26	27	29	35	36	40	41	43	46	47	48	51	54	57	59	61	62	64	67	68	69	...	19	VI
3	6	7	8	10	15	16	18	23	25	26	28	29	31	33	35	36	38	39	40	41	43	48	53	55	57	61	63	64	66	...	20	V
2	6	7	8	10	16	17	18	23	24	26	27	29	31	33	34	36	37	38	41	42	44	49	53	54	56	57	61	65	67	...	21	V
6	7	9	11	15	17	19	20	24	25	28	29	35	37	40	41	44	46	48	49	52	53	57	60	61	63	64	67	69	70	...	22	VI
5	8	9	10	15	18	19	20	24	25	27	28	35	36	41	42	50	51	52	53	54	55	57	58	59	60	63	68	70	71	...	23	VI
6	7	9	10	15	18	20	21	25	26	27	30	35	37	40	41	44	46	48	49	52	55	58	59	61	62	64	65	66	72	...	24	VI
5	8	9	11	15	16	18	19	23	24	27	30	36	37	39	42	45	47	48	50	53	56	58	60	62	64	66	69	71	73	...	25	VI
6	7	11	12	15	16	18	21	23	25	27	29	37	38	40	41	44	48	49	51	54	55	58	61	63	64	66	70	71	75	...	26	VI
-		-		-		-		-		-		-		-		-		-		-		-		-		-		-		...	-	

The following table illustrates rows of the aforementioned closed classes and their linear ordering.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	...
I	0	1	3	4	5	6	7	9	10	11	13	14	16	17	19	20	22	24	26	29	30	32	...
II	1	2	3	5	7	8	9	11	12	13	15	16	18	19	20	21	23	25	27	29	31	32	...
III	1	2	4	6	7	8	12	13	14	15	16	18	19	22	23	24	27	28	29	33	34	35	...
IV	2	3	5	6	7	9	12	13	15	18	19	20	21	24	25	29	30	31	33	37	38	39	...
V	2	6	7	8	10	15	16	17	23	24	25	26	28	30	32	33	35	36	37	40	41	43	...
VI	5	7	9	10	15	16	18	19	23	24	27	28	35	36	39	40	43	46	47	48	51	53	...

An infinite increasing sequence of nonnegative integers $k_{i_1 0}, k_{i_2 1}, \dots, k_{i_n n-1}, \dots$ of the admissible matrix K is said to be a *distributed indicator* of that matrix if the indices $i_j \geq 1$ take the values from an infinite number of different rows, and if between $k_{i_j j-1}$ and $k_{i_{j+1} j}$ there is a jump, then in the basic subgroup B_{i_j} of the group $\overline{B}_{i_j}$ there exists a basic element of order $p^{k_{i_j j-1}+1}$.

For example, if in the aforementioned matrix K we take the elements 0_{10}, 1_{11}, 3_{22}, 4_{23}, 5_{24}, 6_{25}, 7_{26}, 12_{37}, 13_{38}, 14_{39}, $16_{6.10}$, $18_{8.11}$, $22_{9.12}$, $25_{12.13}$, $26_{12.14}$, $33_{13.15}$, $35_{11.16}$, $36_{16.17}$, $47_{17.18}$, $49_{22.19}$, $54_{23.20}$, $55_{23.21}$, $58_{24.22}$, $59_{24.23}$, $62_{25.24}$, $64_{26.25}$, $\dots$ which create a distributed indicator and if under the dots we mean that the admissible matrix K, as $i, j \to \infty$, has positive integers, passing by means of a jump from the element of one row to that of the other row, we find that in the first row following after that element there is a jump.

Definition 2. Let $K = \|k_{ij}\|$, $K' = \|k'_{ij}\|$, $i, j = 0, 1, \dots$, be admissible matrices with respect to the subgroup T, and $k_{0j} \leq k'_{0j}$. We say that $K \leq K'$, if to every row of finite closed class of the matrix K' there corresponds a less or equal row of the closed class of the matrix K, and to the row of an infinite closed class of the matrix K' there corresponds a less or equal row of the closed class of the matrix K, or the distributed indicator. Thus the following conditions are fulfilled:

If the row

$$(k_0, k_1, \dots) \tag{6}$$

of a finite closed class α of the matrix K consists of only nonnegative integers and on that row are mapped the rows of closed classes $\alpha'_i \leq \alpha'_2 \leq \dots$ of the matrix K', where for every i the row of the class α'_i does not likewise contain symbol ∞, while $\underset{i}{U} \alpha'_i$ contains infinite number of rows of the matrix K', then there is an index $n \geq 1$ such that $\overset{n-1}{\underset{i=1}{U}} \alpha'_i$ contains finite number of rows of the matrix K' (assuming the class α_0 does not contain rows), and for the row

$$(k_i', k_2', \dots) \tag{7}$$

of the class α_n' the sequence (6) contains jumps $(k_{t_i}, k_{t_i+1},)$, whereas the sequence (7) contains the numbers k_{m_i}', $i = 1, 2, \dots)$ such that $m_i = 0$,

$$k_{t_i} - t_i \le k_{m_i}' - m_i, \quad t_i - (m_{i+1} - 1) \ge 0 \quad \text{and}$$
$$t_i - m_{i+1} \to \infty \quad \text{as} \quad i \to \infty.$$

It can be easily seen that the relation $\le$ on the set Ω is reflexive and transitive (see [8]), therefore the relation

$$U \,\rho\, V \underset{\text{det}}{=} [U \le V \ \text{ and } \ V \le U]$$

is an equivalence relation on the set Ω, and the relation $\le$

$$\overline{U} \le \overline{V} \underset{\text{det}}{=} U \le V$$

defined on the quotient set $\overline{\Omega} = \Omega/p$ is an ordering.

If $\overline{U}, \overline{V} \in \overline{\Omega}$, where $U = \|u_{ij}\|$ and $V = \|v_{ij}\|$ are admissible matrices, we define the greatest lower bound $\inf(\overline{U}, \overline{V}) = \overline{U} \wedge \overline{V} = \overline{W}$, where $W = \|\min(u_{ij}, v_{ij})\| = \|w_{ij}\|$, $i, j = 0, 1, \dots$. Then the set $\overline{\Omega}$ turns into a meet-semilattice. It is shown in [8] that the function $\Phi : T^{\cdot} \to \overline{\Omega}$, $\overline{\Phi}(a) = \overline{\Phi(a)}$ defined on the group $T^{\cdot}$, where T has the form (3) and Ω is the set of admissible matrices with respect to T, satisfies all the conditions of Theorem 1. Thus the following theorem is valid.

Theorem 2. *The lattice of fully invariant subgroups of a cotorsion hull $T^{\cdot}$, where T is a countable direct sum of torsion-complete p-groups, is isomorphic to the lattice of filters of the semilattice $\overline{\Omega}$.*

As mentioned above, the reduced cotorsion p-adic module has the form (2), where for the sake of simplicity we omit the index p,

$$A = T^{\cdot} \oplus C \tag{8}$$

and assume that the group T has the form (3). Let us now investigate the lattice of fully invariant subgroups of the group (8). Towards this end, we consider the set

$$\Omega^{\cdot} = \overline{\Omega} \cup \overline{H}, \tag{9}$$

where $\overline{\Omega}$ is the semilattice mentioned in Theorem 2, and $\overline{H}$ is the set of increasing sequences of nonnegative integers with only a finite number of jumps, and if between k_i and k_{i+1} there is a jump, then in the basis subgroup B of the group T there is a basis element of order p^{k_i+1}.

As is known, the elements of the set $\overline{\Omega}$ are the classes of admissible matrices, where the first row is defined uniquely. On the set $\Omega^{\cdot}$ we define the relation $\le$. If $\overline{K}$ and $\overline{K}\,' \in \overline{\Omega}$, then $\overline{K} \le \overline{K}\,'$ is defined just as in the set $\overline{\Omega}$. In the remaining cases should be fulfilled the condition $k_i \leq k'_i$, $i = 0, 1, \dots,$ where (k_i), (k'_i) are the first rows of the matrices from the classes of the set $\overline{\Omega}$, or the sequences from $\overline{H}$. The aforementioned relation is, of course, the order relation on the set $\Omega^{\boldsymbol{\cdot}}$.

If $\overline{K}, \overline{K}' \in \overline{\Omega}$, then the greatest lower bound of the elements $\inf(\overline{K}, \overline{K}')$ can be defined in the same way as in the set $\overline{\Omega}$. If $\overline{K} \in \overline{\Omega}$ and $H = (k'_0, k'_1, \dots) \in \overline{H}$ are known, then $\inf(\overline{K}, H) = \min(k_{0j}, k'_j)$, where $(k_{0j})_{j\geq 0}$ is the first row of the admissible matrix K. Taking into account the definition of the matrix K, we can see that $\inf(\overline{K}, H) \in \overline{H}$.

Analogously we define $\inf(H_1, H_2)$, if H_1 and $H_2 \in \overline{H}$. It can be easily seen that this definition satisfies all the requirements of the greatest lower bound. Hence the set $\Omega^{\boldsymbol{\cdot}}$ is a meet-semilattice.

Consider the mapping

$$\Phi^{\boldsymbol{\cdot}} : A \to \Omega^{\boldsymbol{\cdot}} \tag{10}$$

which assigns $\Phi(a)$ to every element $a \in T^{\boldsymbol{\cdot}}$, and if $a = t + c$, where $c \neq 0$, $c \in C$, $t \in T^{\boldsymbol{\cdot}}$, then $\Phi^{\boldsymbol{\cdot}}(a) = H(a)$, where $H(a)$ is the p-indicator of element a. Let us show that the function $\Phi^{\boldsymbol{\cdot}}$ satisfies all five conditions of Mader's theorem.

Condition 1. $\Phi^{\boldsymbol{\cdot}}$ is surjective.

Proof. Let $\overline{K} \in \overline{\Omega}$, where K is an admissible matrix. Then by virtue of [8,Condition 1], there exists $a \in T^{\boldsymbol{\cdot}}$ such that $\overline{\Phi}(a) = \overline{K}$. If $H_1 \in \overline{H}$, $H_1 = (k_0, k_1, \dots)$ and (k_{i_s}, k_{i_s+1}), $s = 1, 2, \dots, n$ are all jumps in H_1, then we denote $k_{i_s} - i_s$ by λ_s. By the condition of the basic subgroup B of the group $T^{\boldsymbol{\cdot}}$, there exist basis element x_s of order $p^{k_{i_s}+1}$. Denote $a_0 = p^{\lambda_1}x_1 + p^{\lambda_2}x_2 + \dots + p^{\lambda_n}x_n$, $a_0 \in T$. Taking into account that $\lambda_1 = k_{i_1} - i_1 = k_0$ and $\lambda_{s+1} = k_{i_{s+1}} - i_{s+1} = k_{i_{s+1}} - (i_s+1)$, the indicator $H(a_0) = (k_0, k_1, \dots, k_{i_n}, \infty, \dots)$. Since C is a reduced algebraically compact torsion free group, there exists $c \in C$ such that the height $h(c) = k_{i_n+1} - (i_n+1)$ (see [2, Corollary 40.4]), then for the elements $a = (a_0, T, T, \dots) \in T^{\boldsymbol{\cdot}}$ and $c \in C$, $a + c \in T^{\boldsymbol{\cdot}} \oplus C$ and $H(a+c) = (k_0, k_1, \dots) = H_1$, i.e., $\Phi^{\boldsymbol{\cdot}}(a+c) = H_1$. □

Condition 2. If $a \in A$ and $f \in \operatorname{End} A$, then $\Phi^{\boldsymbol{\cdot}}(a) \leq \Phi^{\boldsymbol{\cdot}}(fa)$.

Proof. If $a \in T^{\boldsymbol{\cdot}}$, then we can see that $T^{\boldsymbol{\cdot}}$ is a fully invariant subgroup in A, and by virtue of [3, §108], an endomorphism of the group T is uniquely extendable to that of the group $T^{\boldsymbol{\cdot}}$. Consequently, $fa \in T^{\boldsymbol{\cdot}}$, and as is shown in [8], in this case $\Phi^{\boldsymbol{\cdot}}(a) \leq \Phi^{\boldsymbol{\cdot}}(fa)$.

If $a = t + c$, $c \neq 0$, $t \in T^{\boldsymbol{\cdot}}$ then $\Phi^{\boldsymbol{\cdot}}(a) = H(a)$ is the indicator of the element a, not decreasing under the endomorphism, since $h(a) \leq h(fa)$ (see [2, §37]). □

Condition 3. $\Phi^{\boldsymbol{\cdot}}(a+b) \geq \Phi^{\boldsymbol{\cdot}}(a) \wedge \Phi^{\boldsymbol{\cdot}}(b)$.

Proof. Let $a, b \in T^{\cdot}$, then $a + b \in T^{\cdot}$ and $\Phi^{\cdot}(a + b) \geq \Phi^{\cdot}(a) \wedge \Phi^{\cdot}(b)$ (by virtue of [8, Condition 3]). If $a = t + c$, $c \neq 0$, $t \in T^{\cdot}$, $b \in T^{\cdot}$, then $a + b = t + b + c = t' + c$, $t' \in T$. Let $\Phi^{\cdot}(a) = H(a) = (l_0, l_1, \dots) = (\min(t_0, c_0), \min(t_1, c_1), \dots)$, where $H(t) = (t_0, t_1, \dots)$, $H(c) = (c_0, c_0 + 1, \dots)$ (see [2, §37]). Denote $\Phi^{\cdot}(b) = \overline{K}$. $\Phi^{\cdot}(a) \wedge \Phi^{\cdot}(b) = (\min(l_0, k_0), \min(l_1, k_1), \dots)$, where $(k_0, k_1, \dots)$ is the first row of the admissible matrix K. $\Phi^{\cdot}(a + b) = H(t') \wedge H(c)$, but $H(t') = H(t + b) \geq (\min(t_0, k_0), \min(t_1, k_1), \dots)$, therefore $\Phi^{\cdot}(a + b) = H(t') \wedge H(c) \geq (\min(t_0, k_0), \min(t_1, k_1), \dots) \wedge H(c) = \Phi^{\cdot}(a) \wedge \Phi^{\cdot}(b)$.

Assume $a = t + c$ is such as above, and $b = t' + c'$, $t' \in T$, $0 \neq c' \in C$, $H(t') = (t'_0, t'_1, \dots)$, $H(c') = (c'_0, c'_0 + 1, \dots)$. Then $\Phi^{\cdot} \wedge \Phi^{\cdot}(b) = (\min(t_0, c_0), \min(t_1, c_0 + 1), \dots) \wedge (\min(t'_0, c'_0), \min(t'_1, c'_0 + 1), \dots) = (\min(t_0, t'_0, c_0, c'_0), \min(t_1, t'_1, c_0+1, c'_0+1), \dots)$, $\Phi^{\cdot}(a+b) = \Phi^{\cdot}(t+t'+c+c')$. The right-hand side here, if $c+c' \neq 0$, is any class $\overline{M}$ of admissible matrices, or $H(t + t' + c + c')$, otherwise. In the first case too, the first row of the admissible matrix M equals $H(t + t' + c + c')$. But $H(t + t' + c + c') = H(t + t') \wedge H(c + c') \geq (\min(t_0, t'_0), \min(t_1, t'_1), \dots) \wedge (\min(c_0, c'_0), \min(c_0 + 1, c'_0 + 1), \dots) = \Phi^{\cdot}(a) \wedge \Phi^{\cdot}(b)$, which proves Condition 3. □

Condition 4. If $a, b \in A$, $\Phi^{\cdot}(a) \leq \Phi^{\cdot}(b)$, then there exists an endomorphism f of the group A such that $fa = b$.

Proof. In case $a, b \in T^{\cdot}$, the validity of the condition follows from [8, Condition 4]. Let $a = t + c$, $t \in T^{\cdot}$, $0 \neq c \in C$ and let $b = t' \in T^{\cdot}$ and $\Phi^{\cdot}(b) = \overline{K}'$, where the admissible matrix K' has the form I, and its first row is $(k'_{00}, k'_{01}, \dots, k'_{0n}, \dots)$.

Let

$$\Phi^{\cdot}(a) = H(a) = H(t + c) = (k_0, k_1, \dots, k_n, k_{n+1}, k_{n+1} + 1, \dots). \tag{11}$$

For the element $a = t+c$ we assume that $(b_i)_{i \geq 0}$ is a sequence corresponding to the canonical representation $t = (t_0, t_1 + T, \dots)$, $t_i = b_i + pb_{i+1} + p^2 b_{i+2} + \dots$, $i = 0, 1, \dots$, $b_i \in B$, where B is the basic subgroup of the group $T^{\cdot}$. Since equality (11), starting from k_{n+1}, has no jump, the indicator terms $h(a), h(pa), \dots, h(p^n a)$ are obtained from the heights $h(t), h(pt), \dots, h(p^n t)$. Taking into account (4), we can single out from the element $t_0 \in \widehat{T}$ a finite number of summands $t_0^{(1)} = b_0 + pb_1 + \dots + p^k b_k$. Thus $H(b_0^{(1)}) = (k_0, k_1, \dots, k_n, \infty, \dots)$. Obviously, $t_0^{(1)} \in B$ and $t_0 = t_0^{(1)} + t_0^{(2)}$, $t_0^{(2)} = p^{k+1} b_{k+1} + \dots \in \widehat{T}$, whence $t = (t_0, t_1 + T, \dots) = (t_0^{(1)}, T, \dots) + (t_0^{(2)}, t_1 + T, \dots)$, where $h(t_0^{(2)}, t_1 + T, \dots) \geq k_{n+1} - (n + 1)$.

Analogously, let $t' = (t'_0, t'_1 + T \dots)$ and $(b'_i)_{i \geq 1}$ be its corresponding sequence of the basic subgroup B. Then $t'_i = b'_i + pb'_{i+1} + \dots$, $i = 0, 1, \dots$. Since by the condition, $\Phi^{\cdot}(a) = H(a) = (k_0, k_1, \dots, k_n, k_{n+1}, k_{n+1}+1, \dots) \leq$

$H(t') = (k'_{00}, k'_{01}, \dots)$, therefore $t'_0 = b'_0 + pb'_1 + \dots$ can be represented in the form $t'_0 = t_0^{'(1)} + t_0^{'(2)}$, where $t_0^{'(1)} = b'_0 + pb'_1 + \dots + p^m b'_m$, $t_0^{'(2)} = p^{m+1} b'_{m+1} + \dots$ and $H(t_0^{'(1)} = (k'_{00}, \dots, k'_{0n}, \infty, \dots)$. Obviously, $t_0^{'(1)} \in B$. Then $t' = (t_0^{'(1)}, T, \dots) + (t_0^{'(2)}, t'_1 + T, \dots)$ and $h(t_0^{'(2)}, t'_1 + T, \dots) \geq k_{n+1} - (n+1)$. We have $H(t_0^{1)}) \leq H(t_0^{'(1)})$, $t_0^{(1)}, t_0^{'(1)} \in B$ and since the separable subgroup B is fully transitive (see [3, §65]), there exists its endomorphism $f_0^{(1)}$ defined by a finite number of elements $b_0, b_1, \dots, b_k \in B$ which transfer $t_0^{(1)}$ to $t_0^{'(1)}$. It is evident that the elements $b_0, b_1, \dots, b_k$ are defined by means of a finite number of elements (x_α). For the remaining basic elements x_β we put $f_0^{(1)} x_\beta = 0$. Thus we obtain a uniquely defined endomorphism (see [3, §108]) of the group $T^{\cdot}$ for which $f_0^{(1)} t_0^{(1)} = t_0^{'(1)}$, and $f_0^{(1)} t_0^{(2)} = (t_0^*, T, \dots) \in B \subset T$, $h(t_0^*) \geq k_{n+1} - (n+1)$. Obviously, $f_0^{(1)} t_i \in B \subset T$, $i = 1, 2, \dots$.

We have $h(c) = (k_{n+1} - (n+1)) \leq h((t_0^{'(2)}, t'_1 + T, t'_2 + T, \dots) - t_0^*)$. Since C is an algebraically compact torsion free p-adic module, $c \in C$ can be embedded into the direct summand J_p (see [2, §40.4, §51.1]), and there exists $f \in \mathrm{End} A$ such that $fc = (t_0^{'(2)}, t'_1 + T, t'_2 + T, \dots) - (t_0^*, T, \dots)$ and $f|T^{\cdot} = f_0^{(1)}$ on the direct summand $T^{\cdot}$. Then

$$\begin{aligned} fa &= f(t+c) = ft + fc = f\big((t_0^{(1)}, T, \dots) + (t_0^{(2)}, t_1 + T, \dots)\big) + \\ &+ (t_0^{'(2)}, t'_1 + T, \dots) - (t_0^*, T, \dots) = f_0^{(1)}\big((t_0^{(1)}, T, \dots)\big) + \\ &+ f_0^{(1)}\big((t_0^{(2)}, t_1 + T, \dots)\big) + (t_0^{'(2)}, t'_1 + T, \dots) - (t_0^*, T, \dots) = \\ &= (t_0^{'(1)}, T, \dots) + (t_0^*, T, \dots) + (t_0^{'(2)}, t'_1 + T, \dots) - (t_0^*, T, \dots) = \\ &= (t'_0, t'_1 + T, \dots) = b. \end{aligned}$$

It is not difficult to see that in the same way as above we can consider the cases for $a = t + c$, or $a = c$, $b = t_1 + c_1$, or $b = c_1$, $t, t_1 \in T^*$, $c, c_1 \in C$. □

Condition 5. If G is a fully invatiant subgroup in A and $a, b \in G$, then there exists $c \in G$ such that $\Phi^{\cdot}(c) = \Phi^{\cdot}(a) \wedge \Phi^{\cdot}(b)$.

Proof. Since G is a fully invariant subgroup of the group A, by virtue of [2. §9],

$$G = (G \cap T^{\cdot}) \oplus (G \cap C). \tag{12}$$

As we mentioned when proving Condition 2, $T^{\cdot}$ is a fully invariant subgroup in A. Therefore $G \cap T^{\cdot}$ is a fully invariant subgroup of the group A. Let $a = t_1 + c_1$, $b = t_2 + c_2$ and at least one of $c_1, c_2 \neq 0$. Then by (12), $t_1, t_1 \in G \cap T^{\cdot}$, $c_1, c_2 \in G \cap C$. According to [8, Condition 5], there exists $t \in G \cap T^{\cdot}$ such that $\Phi^{\cdot}(t) = \Phi^{\cdot}(t_1) \wedge \Phi^{\cdot}(t_2)$. c_1 and c_2 are elements of infinite order from C, therefore $H(c_1) \wedge H(c_2)$ is equal either to $H(c_1)$ or to $H(c_2)$.

If it is $H(c_1)$, then $t + c_1 \in G$ and $\Phi^{\cdot}(t + c_1) = H(t) \wedge H(c_1) = H(t_1) \wedge H(t_2) \wedge H(c_1) \wedge H(c_2) = (H(t_1) \wedge H(c_1)) \wedge (H(t_2) \wedge H(c_2)) = \Phi^{\cdot}(a) \wedge \Phi^{\cdot}(b)$. If both c_1 and c_2 are equal to zero, then the condition will be fulfilled by virtue of [8, Condition 5]. $\square$

As we mentioned concerning the equality (8), the group T has the form (3) and for the sake of simplicity we omitted the index p. The same can be done regarding (9) and (10). As far as our next step is to consider the direct sum of torsion-complete groups, we revert to the index p and formulate the obtained results.

Thus we have found that the function $\Phi^{\cdot}_p : A_p \to \Omega^{\cdot}_p$, where $A_p = T^{\cdot}_p \oplus C_p$, and T_p is of the form (3), satisfies all five conditions of Mader's theorem. Thus the following theorem is valid.

Theorem 3. *The lattice of fully invariant submodules of the reduced cotorsion p-adic module A_p, whose torsion part is a countable direct sum of torsion-complete p-adic modules, is isomorphic to the lattice of filters of the semilattice $\Omega^{\cdot}_p$.*

Corollary 1. *The lattice of fully invariant subgroups of a reduced cotorsion group A_p, whose torsion part is a countable direct sum of torsion-complete p-groups, is isomorphic to the lattice of filters of the semilattice $\Omega^{\cdot}_p$.*

As mentioned above, a reduced cotorsion group A has the form (1), where $T = \underset{p}{\oplus} T_p$ is the torsion part of the group A.

A torsion group T_i is said to be torsion-complete if its p-component T_{ip} is a torsion-complete p-group for every p. $T_i = \underset{p}{\oplus} T_{ip}$. Let T be a countable direct sum of torsion-complete groups

$$T = \overset{\infty}{\underset{i=1}{\oplus}} T_i = \overset{\infty}{\underset{i=1}{\oplus}} \Big(\underset{p}{\oplus} T_{ip} \Big) = \underset{p}{\oplus} \Big(\overset{\infty}{\underset{i=1}{\oplus}} T_{ip} \Big).$$

Then T us such a torsion group whose every p-component is a countable direct sum of torsion-complete p-groups. Consequently, if the torsion part T of a reduced cotorsion group A is a countable direct sum of torsion-complete groups, then it will be of the form (1), where every T_p is a direct sum of torsion-complete p-groups, and C_p is an algebraically compact torsion free group.

Obviously, for every prime number p there exists a semilattice Ω_p described in Theorem 3, which corresponds to the group $T^{\cdot}_p \oplus C_p$.

Consider the set

$$\Omega^{\cdot} = \prod_p \Omega^{\cdot}_p.$$

For its elements $\alpha = (\dots, \alpha_p, \dots)$ and $\beta = (\dots, \beta_p, \dots)$ we put $\alpha \le \beta \Leftrightarrow \alpha_p \le \beta_p$ and $\alpha \wedge \beta = (\dots, \alpha_p \wedge \beta_p, \dots)$ for all $\alpha_p, \beta_p \in \Omega^{\cdot}_p$. Thus the set

$\Omega^{\cdot}$ turns into a meet-semilattice. As mentioned above, every group $T^{\cdot}_p \oplus C_p$ is at the same time a reduced cotorsion p-adic module, therefore it is q-divisible, where $q \neq p$ is a prime number. Indeed, for every natural number n, q^n is a p-adic unit (see [2, §3]), hence there exists $q^{-n} \in Q^*_p$, and for all $a \in T^{\cdot}_p \oplus C_p$, $a = q^n \cdot q^{-n}a = q^n(q^{-n}a) = q^n b$, $b \in Q^*_p$; i. e. a is divisible by q^n.

On the other hand, any endomorphism f of the ring of endomorphisms $E(A)$ of the group A can be written in the form $(\dots, f_p, \dots)$, where $f_p \in E(A_p)$. Indeed, since every A_p, $q \neq p$ is divisible, therefore A_p is a fully invariant subgroup in A, and $A/\oplus A_p$ is a divisible group. Consider the mapping

$$\alpha : E(A) \to E(A_2) \times E(A_3) \times \dots \times E(A_p) \times \dots, \quad \alpha f = (\dots, f_p, \dots),$$

where $f_p = f|A_p$ is restriction of f on A_p. Obviously, α is an endomorphism. If $f \neq g$ and $\alpha f = \alpha g$, $f, g \in E(A)$, then by the definition of f_p and g_p, $\underset{p}{\oplus} A_p \subset \operatorname{Ker}(f-g)$ and $\operatorname{Im}(f-g) \cong A/\operatorname{Ker}(f-g)$ is a divisible group, since $A/\underset{p}{\oplus} A_p$ is divisible. On the other hand, A is a reduced group, hence $f - g = 0$, i. e. $f = g$. Thus α is an isomorphism. Consequently, any endomorphism f of the group A can be written as follows: $f = (\dots, f_p, \dots)$.

Thus after all the above remarks we can easily see that since for every p the function $\Phi_p : A_p \to \Omega^{\cdot}_p$ satisfies the conditions of Theorem 1, the function $\Phi : A \to \Omega^{\cdot}$, $\Phi(a) = (\dots, \Phi_p(a_p), \dots)$, $a_p \in A_p$ likewise satisfies these conditions. Thus the following theorem is valid.

Theorem 4. *If the torsion part of a reduced cotorsion group A is a countable direct sum of torsion-complete groups, then the lattice of fully invariant subgroups of the group A is isomorphic to the lattice of filters of the semilattice $\Omega^{\cdot}$.*

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