Fully transitivity of abelian groups

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Abstract

We study fully transitive abelian groups, their direct sums and direct products.

Introduction

Fully transitivity is one of interesting properties of abelian groups (an abelian group A is said to be fully transitive if for all elements $a, b \in A$ for which $\mathbb{H}(a) \leqslant \mathbb{H}(b)$ there exists an endomorphism of the group A mapping a into b). This concept was introduced by Kaplansky in [17] for reduced modules over a complete discrete valuation ring and for abelian p-groups. He proved that any reduced separable p-group is fully transitive. Furhter, Kaplansky puts a question: is an arbitrary abelian p-group fully transitive? First negative examples were presented by Megibben [20]. Hill [14] demonstrated that totally projective p-groups introduced by Nunke [23] are fully transitive. Corner [5] considered the following concept: let Φ be a subring of the ring $\mathrm{E}(G)$ and H be a Φ-invariant subgroup of a reduced p-group G, then Φ is fully transitive on H if for arbitrary elements $x, y \in H$ such that $U_G(x) \leqslant U_G(y)$ there exists $\phi \in \Phi$ with the property $\phi x = y$. He proved that a p-group G is fully transitive if and only if $\mathrm{E}(G)$ is fully transitive on $p^\omega G$ and presented an example of a reduced p-group which is not fully transitive.

Let λ be the limit ordinal. An abelian p-group G is said to be λ-separable (le Borgne [16]) if each its finite system of elements is contained in a certain direct summand of the group G and this summand is a totally projective group of length less than λ. Le Borgne proves that each λ-separable group is fully transitive.

Hill and Megibben [15] described IT-groups (they say that a p-group G is an IT-group if it is isomorphic to an isotype subgroup of a totally projective group); in particular, they proved that all IT-group are fully transitive.

Krylov [36], by analogy with fully transitive p-groups, considered the concept of fully transitivity for torsion free groups. Arnold [1] described homogeneous fully transitive torsion free finite rank groups. Krylov in [37, 40] presented the description of fully transitive countable homogeneous torsion free groups and fully transitive torsion free groups whose p-rank (i.e., rank of the group G/pG) is finite. This generalized the results obtained by Dobrusin [34]. Krylov considered non-homogeneous fully transitive torsion free groups in [38, 40], and in [39] he constructed an example of super-decomposable fully transitive groups. Chekhlov [50] obtained a characterization of fully transitive torsion free groups all non-zero endomorphisms of which are monomorphisms.

An important subclass of the class of fully transitive groups is the class of quasi-pure injective groups (*qpi*-groups), i.e., groups A such that all homomorphisms of each pure subgroup B in A can be extended to an endomorphism of the group A. The study of *qpi*-groups is announced as Problem 17a in the book [46]. Torsion full groups, whose structure is known, are reduced torsion *qpi*-groups. Torsion free *qpi*-groups were described by Chekhlov [48, 49]. Mixed *qpi*-groups were studied by Dobrusin [33].

The case of mixed fully transitive groups was first considered in Mader's work [19], where he proves that reduced p-adic algebraically compact groups are fully transitive.

Moskalenko [44] studied the group $\mathrm{Ext}(\mathbb{Z}(p^\infty), T)$ and demonstrated that it is fully transitive in the case when T is a direct sum of cyclic p-groups.

Fully transitive p-groups were also considered in papers by Griffith [11], Meggiben [21], Carroll and Goldsmith [4], Files and Goldsmith [10], Paras and Strüngmann [24]. Torsion free fully transitive were investigated by Grinshpon [29], Hausen [12], Dugas and Hausen [7], Dugas and Shelah [8]. Fully transitive torsion free modules were studied in Files' work [9]. Hennecke and Strüngmann [13] considered fully transitive p-local modules. Fully transitive separable abelian groups and their direct products are characterized in the paper of Grinshpon and Misyakov [32]. Fully transitivity of direct sums and direct products of arbitrary abelian groups was considered in the papers of Grinshpon and Misyakov [31], Misyakov [42]. The work by Grinshpon [30] presents the description of fully invariant subgroups and their lattices for fully transitive groups from different classes of abelian groups and the properties of fully transitive groups which are k-direct sums of abelian groups. A survey of results on fully transitive groups and groups close to them can found in [28]. Some problems on fully transitive abelian groups are mentioned in the book [18]. In particular, Problem 45 is stated as follows: "Find necessary and sufficient conditions for fully transitivity of the direct sum $\sum_{i\in I}^{\oplus} G_i$ and product $\prod_{i\in I} G_i$ of groups G_i".

This paper presents main results obtained by the author on fully transitive abelian groups in different years and connected with the above-stated problem. In the first section, we present necessary and sufficient conditions for fully transitivity of direct sums of arbitrary abelian groups and sufficient conditions for fully transitivity of direct products of such groups. In the second section we describe fully transitive direct products of generalized separable torsion free groups. In the third section, fully transitivity of direct products of s-generalized slender groups is studied. In the fourth section we present a complete description of fully transitivity of direct products of arbitrary separable groups. We investigate influence of fully transitivity on splitability of direct products of bf-generalized slender groups and on separability of direct products of groups.

The proofs are significantly revised and standardized.

The designations are as follows. $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the group of integers; $\mathbb{Z}(p^\infty)$ is the quasicyclic group; π is the set of all prime numbers; $o(a)$ is the order of the element a; $h_p(a)$ $(h_p^*(a))$ is the height (generalized height) of the element a; $H_p(a)$ is the Ulm sequence of the element a; $\chi_A(a)$ or $\chi(a)$ is the characteristic of the element a in a torsion free group A; $\mathbb{H}_A(a)$ or $\mathbb{H}(a)$ is the height matrix of the element a in the group A; $\mathrm{E}\,(A)$ is the endomorphism ring of the group A, $\mathrm{Hom}(A,\, B)$ is the group of homomorphisms from the group A to the group B; $t_A(a)$ or $t(a)$ is the type of the element a in a torsion free group A; $t(A)$ is the type of a homogeneous torsion free group; $\pi(A)$ is the set of prime numbers p such that $pA \neq A$; $T(A)$ is the torsion subgroup of the group A; $T_p(A)$ is the p-component of the group A. Designations and terms that are used without explanation are standard and taken from [46, 47]. All the groups are supposed to be abelian.

§1. Fully transitivity of direct sums and direct products of abelian groups

In this section, a criterion for fully transitivity of direct sums of abelian groups is established and sufficient conditions for fully transitivity of direct products of arbitrary reduced abelian groups are presented. An example demonstrates existence of fully transitive direct products of reduced groups for which one of sufficient conditions for fully transitivity is not satisfied.

Let $p_1, p_2, \dots, p_n, \dots$ be a sequence of all prime numbers in their increasing order. We recall that $h_p^*(a) = \sigma$ (where σ is an ordinal), if $a \in p^\sigma A \backslash p^{\sigma+1} A$. In the case $a \in p^\tau A = p^{\tau+1}$, we, as usual, put $h_p^*(a) = \infty$ and suppose that ∞ is more than any ordinal. The matrix of dimensions $\omega \times \omega$ whose elements are ordinal numbers or a symbol ∞ is said to be the height matrix. Such a matrix

$$(\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

can be interpreted as a function $f : \mathbb{N} \times \mathbb{N} \to \mathfrak{O}n \cup \{\infty\}$ (where $\mathfrak{O}n$ is the class of all ordinals) such that $f(i,j) = \alpha_{ij}$. The class of all height matrices is denoted by $\mathfrak{M}$. Let $M_1, M_2 \in \mathfrak{M}$, $M_1 = (\alpha_{ij})$, $M_2 = (\beta_{ij})$, then we say that $M_1 \le M_2$ if $\alpha_{ij} \le \beta_{ij}$ for all $i, j \in \mathbb{N}$. If $M' = \{(\alpha_{ij}^{(r)}) \mid r \in K\}$ is a set of height matrices, we naturally define $\inf_{\mathfrak{M}} M'$, namely, $\inf_{\mathfrak{M}} M' = (\beta_{ij})$, where β_{ij} is the least of the elements $\alpha_{ij}^{(r)}$, $r \in K$. An element a of the group A can be connected with a height matrix

$$\mathbb{H}(a) = \begin{pmatrix} h_{p_1}^*(a) & h_{p_1}^*(p_1 a) & \dots & h_{p_1}^*(p_1^k a) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ h_{p_n}^*(a) & h_{p_n}^*(p_n a) & \dots & h_{p_n}^*(p_n^k a) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = (\sigma_{ij}).$$

The nth string of the matrix $\mathbb{H}(a)$ is said to be the p_n-indicator of the element a and denoted by $H_{p_n}(a)$.

Fuchs [47], following Kaplansky [17], calls a reduced abelian p-group a fully transitive group if for all its elements a and b for which $H_p(a) \le H_p(b)$ there exists an endomorphism φ mapping a to b. We can introduce the concept of fully transitivity for an arbitrary abelian group by analogy, i.e., we can call an abelian group G fully transitive if for all elements a, $b \in G$

such that $\mathbb{H}(a) \leq \mathbb{H}(b)$ there exists $\varphi \in \mathrm{E}(G)$ with the property $\varphi(a) = b$. However, even for rather simple non-reduced groups, we obtain that they are not fully transitive in the sense of this definition.

Let us consider a simple example. Here, as it is commonly accepted [46, 47], we suppose that the generalized p-height of the zero element coincides with its usual p-height and equals ∞.

Example 1.1. Let $G =< a > \oplus \mathbb{Z}(p_n^\infty)$, where $o(a) = p_n^s$ and $b \in \mathbb{Z}(p_n^\infty)$, $o(b) = p_n^k$, $k > s$. We have $\mathbb{H}(a) < \mathbb{H}(b)$, because $H_{p_n}(a) = (0, 1, \dots, s-1, \infty, \dots)$, $H_{p_n}(b) = (\infty, \dots)$ and $H_{p_i}(a) = H_{p_i}(b) = (\infty, \dots)$ for all $i \neq n$. However, there are no $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = b$, because an endomorphism does increase order of an element.

Definition 1.1. (Grinshpon [31]) *A group G is said to be fully transitive if for all $a, b \in G$ the condition $\mathbb{H}(a) \leq \mathbb{H}(b)$ and $o(a) \,\vdots\, o(b)$ (order of the element a is divided by that of the element b) implies existence of $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = b$. Here we suppose that if $o(a) = \infty$, the condition $o(a) \,\vdots\, o(b)$ is satisfied for each $b \in G$.*

Proposition 1.1. (Grinshpon [31]) *For a reduced group G, the following are equivalent:*

1) for all $a, b \in G$, the condition $\mathbb{H}(a) \leq \mathbb{H}(b)$ and $o(a) \,\vdots\, o(b)$ implies existence of $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = b$;
2) for all $a, b \in G$, the condition $\mathbb{H}(a) \leq \mathbb{H}(b)$ implies existence of $\varphi \in E(G)$ such that $\varphi(a) = b$.

Let's demonstrate that the question about fully transitivity of a group can be considered as a corresponding question for its reduced subgroup.

Theorem 1.2. *A group G is fully transitive if and only if its reduced subgroup is fully transitive.*

Proof. If G is a fully transitive group, fully transitivity of its reduced subgroup easily follows from Definition 1.1. Conversely. Let $G = A \oplus B$, where A is the reduced subgroup of the group G, B is its divisible subgroup. Consider arbitrary non-zero elements $a, b \in G$ such that $\mathbb{H}(a) \leq \mathbb{H}(b)$ and $o(a) \,\vdots\, o(b)$. Let $a = a_1 + b_1$, $b = a_2 + b_2$ and 1) $a_1 \neq 0$, $a_2 \neq 0$, where $a_1, a_2 \in A$; $b_1, b_2 \in B$. Then $\mathbb{H}(a) = \mathbb{H}(a_1)$, $\mathbb{H}(b) = \mathbb{H}(a_2)$ and $\mathbb{H}(a_1) \leq \mathbb{H}(a_2)$. So, existence of $\varphi \in \mathrm{E}(A)$ such that $\varphi(a_1) = a_2$ follows from fully transitivity of the group A. Now we can extend φ to an endomorphism $\psi \in \mathrm{E}(G)$, supposing $\psi(c) = \varphi(c)$ if $c \in A$, and $\psi(c) = 0$ if $c \in B$.

2) $a_1 \neq 0$, $a_2 = 0$ and let $o(a) < \infty$ (the case $o(a) = \infty$ is proved similarly). Consider the subgroup $< a > \subset G$. Construct a mapping $\varphi :< a > \longrightarrow B$ by the following rule: $\varphi(a) = b$. It is defined correctly because if $ma = 0$ for a certain $m \in \mathbb{N}$, then $m \vdots o(a)$ and, consequently, $m \vdots o(b)$. Then $\varphi(0) = \varphi(ma) = m\varphi(a) = mb = 0$. It is easy to verify that φ is a homomorphism. Let $\rho :< a > \longrightarrow G$ be an embedding. Since B is an injective group, there exists $\psi \in \mathrm{Hom}\,(G, B)$ such that $\psi\rho = \varphi$, i.e., $\psi(a) = b$.

3) The case $a_1 = 0$, $a_2 \neq 0$ is impossible.

4) Let $a_1 = 0$, $a_2 = 0$, i.e., $a, b \in B$ and $o(a) \vdots o(b)$. Consider the subgroups $< a >$, $< b >$ from B. There exists a homomorphism $\varphi :< a > \longrightarrow < b >$ such that $\varphi(a) = b$. Let $i :< a > \longrightarrow B$ be an embedding. Since B is an injective group, $\psi i = \varphi$ for a certain $\psi \in \mathrm{E}\,(B)$, and $\psi(a) = b$. The endomorphism ψ evidently can be extended to an endomorphism of the group G. Thus, the group G is fully transitive.

As follows from Theorem 1.2, fully transitivity of groups can be studied by use of only reduced groups. So, in the sequel, by a group we mean a reduced groups.

We give two main concepts which will be used in studying fully transitivity of direct sums and direct products. They were first formulated by S. Ya. Grinshpon [29] for torsion free groups. Later and independently the first of them appeared in the work of Files and Goldsmith [10].

Definition 1.2. *A system of groups* $\{G_i\}_{i\in I}$ *is said to be fully transitive if for each pair of groups* $(G_i, G_j)_{i,\,j\in I}$ *(i can coincide with j) the following condition is satisfied: from* $a \in G_i$, $b \in G_j$ *and* $\mathbb{H}(a) \leq \mathbb{H}(b)$ *follows existence of* $\varphi \in \mathrm{Hom}(G_i, G_j)$ *with the property* $\varphi(a) = b$.

Definition 1.3. *We say that a system of groups* $\{G_i\}_{i\in I}$ *satisfies the monotonicity condition for height matrices if for each group* G_j *and each element* $0 \neq a_j \in G_j$ $(j \in I)$ *the relations*

1) $\inf_{\mathfrak{M}}\{\mathbb{H}(a_{i_1}), \ldots, \mathbb{H}(a_{i_s})\} \leq \mathbb{H}(a_j)$, *where* $a_{i_k} \in G_{i_k}$, $i_k \in I$, $k = \overline{1,\, s}$, $i_k \neq i_l$, $k \neq l$;

2) $\mathbb{H}(a_j) \ngeq \mathbb{H}(a_k)$ *for all* $k = \overline{1,\, s}$,

imply existence of the elements $a_{j_1}, \ldots, a_{j_r} \in G_j$ *with the following properties:*

1') $a_{j_1} + \ldots + a_{j_r} = a_j$;
2') for each element a_{j_l} $(l = \overline{1,\, r})$ there exists an element a_{i_k} $(k = \overline{1,\, s})$ such that $\mathbb{H}(a_{j_l}) \geq \mathbb{H}(a_{i_k})$.

Instead of the phrase "monotonicity condition for height matrices" we'll often write simply "monotonicity condition". It is easy to show that if a system of groups is fully transitive, each its subsystem is also fully transitive. If a system of groups satisfies the monotonicity condition, each its subsystem also satisfies the condition.

Theorem 1.3. *A group $G = \bigoplus\limits_{i \in I} G_i$ is fully transitive if and only if the system of groups $\{G_i\}_{i \in I}$ is fully transitive and satisfies the monotonicity condition.*

Proof. Let G be a fully transitive group. Consider arbitrary groups $G_\alpha,\, G_\beta \in \{G_i\}_{i \in I}$ and elements $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$ such that $\mathbb{H}(g_\alpha) \leq \mathbb{H}(g_\beta)$. Then $\mathbb{H}(\rho_\alpha g_\alpha) \leq \mathbb{H}(\rho_\beta g_\beta)$, where $\rho_\alpha : G_\alpha \longrightarrow G$, $\rho_\beta : G_\beta \longrightarrow G$, are embeddings. Fully transitivity of the group G implies existence of $\varphi \in \mathrm{E}\,(G)$ such that $(\varphi \rho_\alpha)(g_\alpha) = \rho_\beta(g_\beta)$ and, therefore, $(\pi_\beta \varphi \rho_\alpha)(g_\alpha) = g_\beta$, where $\pi_\beta : G \longrightarrow G_\beta$ is a projection.

Let's demonstrate that the system $\{G_i\}_{i \in I}$ satisfies the monotonicity condition. Consider an arbitrary group $G_j \in \{G_i\}_{i \in I}$ and an element $0 \neq a_j \in G_j$ such that $\inf_{\mathfrak{M}}\{\mathbb{H}(a_{i_1}), \ldots, \mathbb{H}(a_{i_s})\} \leq \mathbb{H}(a_j)$, $\mathbb{H}(a_j) \ngeq \mathbb{H}(a_k)$ for all $k = \overline{1,\, s}$. For each $k = \overline{1,\, s}$, consider embeddings $\rho_{i_k} : G_{i_k} \longrightarrow G$ and $\rho_j : G_j \longrightarrow G$, where $G_{i_k} \in \{G_i\}_{i \in I}$ and $a_{i_k} \in G_{i_k}$. Then $\mathbb{H}(\rho_{i_1}(a_{i_1}) + \ldots + \rho_{i_s}(a_{i_s})) \leq \mathbb{H}(\rho_j(a_j))$, and fully transitivity of the group G implies existence of $\varphi \in \mathrm{E}\,(G)$ such that $\rho_j(a_j) = \varphi(\rho_{i_1}(a_{i_1}) + \ldots + \rho_{i_s}(a_{i_s})) = (\varphi\rho_{i_1})(a_{i_1}) + \ldots + (\varphi\rho_{i_s})(a_{i_s})$. We have $a_j = (\pi_j \rho_j)(a_j) = (\pi_j \varphi \rho_{i_1})(a_{i_1}) + \ldots + (\pi_j \varphi \rho_{i_s})(a_{i_s})$, where $\pi_j : G \longrightarrow G_j$ is a projection. Let $a_{j_k} = (\pi_j \varphi \rho_{i_k})(a_{i_k})$ for all $k = \overline{1,\, s}$. Then $a_j = a_{j_1} + \ldots + a_{j_s}$, and for each element a_{j_k} $(k = \overline{1,\, s})$ there exists an element a_{i_k} $(k = \overline{1,\, s})$ such that $\mathbb{H}(a_{i_k}) \leq \mathbb{H}(a_{j_k})$. Conversely. Let $a,\, b \in G$, and $\mathbb{H}(a) \leq \mathbb{H}(b)$. Let $a = a_{i_1} + \ldots + a_{i_r}$, $b = b_{l_1} + \ldots + b_{l_n}$, where $a_{i_t} \in G_{i_t}$, $b_{l_j} \in G_{l_j}$, $i_t,\, l_j \in I$ for all $t = \overline{1,\, r}$ and $j = \overline{1,\, n}$. Let's show that for each element b_{l_j} $(j = \overline{1,\, n})$ there exists $\varphi_j \in \mathrm{Hom}(G,\, G_{l_J})$ such that $\varphi_j(a) = b_{l_j}$. Then $b = (\varphi_{j_1} + \ldots + \varphi_{j_n})(a)$ and $\varphi_{j_1} + \ldots + \varphi_{j_n} \in \mathrm{E}\,(G)$. If for the element b_{l_j} there exists an element a_{i_k} such that $\mathbb{H}(b_{l_j}) \geq \mathbb{H}(a_{i_k})$, existence of $\psi_j \in \mathrm{Hom}(G_{i_k},\, G_{l_j})$ with the property $\psi_j(a_{i_k}) = b_{l_j}$ follows from fully transitivity of the system $\{G_i\}_{i\in I}$. Then we put $\varphi_j(a) = \psi_j(a_{i_k})$, i.e., $\varphi_j = \psi_j\pi_{i_k}$, where $\pi_{i_k} : G \longrightarrow G_{i_k}$ is a projection. Let there be no elements a_{i_k} with the property $\mathbb{H}(b_{l_j}) \geq \mathbb{H}(a_{i_k})$. Since $\mathbb{H}(b_{l_j}) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a_{i_k})\}_{k=\overline{1,r}}$ and the system $\{G_i\}_{i\in I}$ satisfies the monotonicity condition, there exist elements $b_{l_{j_\tau}} \in G_{l_j}$ $(\tau = \overline{1,\, t})$ such that $b_{l_j} = b_{l_{j_1}} + \ldots + b_{l_{j_t}}$, and for each $b_{l_{j_\tau}}$ $(\tau = \overline{1,\, t})$ one can find $a_{i_{k_\tau}} \in \{a_{i_k}\}_{k=\overline{1,r}}$ such that $\mathbb{H}(b_{l_{j_\tau}}) \geq \mathbb{H}(a_{i_{k_\tau}})$. Then fully transitivity of the system $\{G_i\}_{i\in I}$ implies existence of $\psi_{j_\tau} \in \mathrm{Hom}(G_{i_{k_\tau}}, G_{l_j})$ such that $\psi_{j_\tau}(a_{i_{k_\tau}}) = b_{l_{j_\tau}}$, and $\psi_{j_1}(a_{i_{k_1}}) + \ldots + \psi_{j_t}(a_{i_{k_t}}) = b_{l_j}$. Then $\varphi_j(a) = \psi_{j_1}(a_{i_{k_1}}) + \ldots + \psi_{j_t}(a_{i_{k_t}})$, where $\varphi_j = \psi_{j_1}\pi_{i_{k_1}} + \ldots + \psi_{j_t}\pi_{i_{k_t}}$, $\pi_{i_{k_\tau}} : G \longrightarrow G_{i_{k_\tau}}$ are projections. Thus, $\varphi_j(a) = b_{l_j}$.

The following Proposition demonstrates that fully transitivity and fulfilment of the monotonicity condition for the system of groups $\{G_i\}_{i\in I}$ are necessary for fully transitivity of a direct product of arbitrary groups.

Proposition 1.4. *If $G = \prod\limits_{i\in I} G_i$ is a fully transitive group, the system $\{G_i\}_{i\in I}$ is fully transitive and satisfies the condition of monotonicity.*

Proof of this statement is similar to proof of necessity in the previous theorem.

The following Definition will be useful to elucidate some sufficient conditions for fully transitivity of the group $G = \prod\limits_{i\in I} G_i$.

Definition 1.4. *We say that a system of groups $\{G_i\}_{i\in I}$ satisfies the condition of finiteness for height matrices, if for each group $G_j \in \{G_i\}_{i\in I}$ and each element $0 \neq g_j \in G_j$ such that $o(g_j) = \infty$ the conditions*

$$\mathbb{H}(g_j) \geq \inf{}_{\mathfrak{M}}\{\mathbb{H}(g_\gamma)\}_{\gamma\in J},\ \textit{where}\, |J| = \aleph_0\ J \subseteq I,$$

imply existence of a finite subsystem of elements $\{g_{\gamma_k}\}_{\gamma_k\in J,\, k=\overline{1,n}}$ such that

$$\mathbb{H}(g_j) \geq \inf{}_{\mathfrak{M}}\{\mathbb{H}(g_{\gamma_k})\}_{\gamma_k\in J,\, k=\overline{1,n}}\,.$$

Sometimes, instead of the term "condition of finiteness for height matrices" we write simply "condition of finiteness".

Proposition 1.5. *A group* $G = \prod\limits_{i\in I} G_i$ *is fully transitive, if the system of groups* $\{G_i\}_{i\in I}$ *is fully transitive and satisfies the conditions of monotonicity and finiteness.*

Proof. Let $a,\, b \in G$ and $\mathbb{H}(a) \le \mathbb{H}(b)$, where $a = (\dots, a_i, \dots)$, $b = (\dots, b_i, \dots)$. Let's demonstrate that for an arbitrary coordinate b_j of the element b there exists a finite subsystem of coordinates $\{a_{i_l}\}_{l=\overline{1,\,n}}$ of the element a such that $\mathbb{H}(b_j) \ge \inf_{\mathfrak{M}}\{\mathbb{H}(a_{i_l})\}_{l=\overline{1,\,n}}$. Consider two cases: 1) $o(b_j) < \infty$, 2) $o(b_j) = \infty$, here we assume that almost all the coordinates a_i of the element a are different from zero. Let $o(b_j) < \infty$. Since the height matrix $\mathbb{H}(b_j)$ contains a finite number σ_{nk} of ordinals different from ∞ and $\mathbb{H}(b_j) \ge \inf_{\mathfrak{M}}\{\mathbb{H}(a_i)\}_{i\in I}$, there exists a finite subset $\{a_{i_l}\}_{l=\overline{1,\,n}} \subset \{a_i\}_{i\in I}$ such that $\mathbb{H}(b_j) \ge \inf_{\mathfrak{M}}\{\mathbb{H}(a_{i_l})\}_{l=\overline{1,\,n}}$. If $o(b_j) = \infty$, then, taking into account the inequality $\mathbb{H}(b_j) \ge \mathbb{H}(a)$ and the fact that the height matrix of every element of a group consists of at most countable set of ordinals, we prove existence of the subset $\{a_\tau\}_{\tau\in J} \subseteq \{a_i\}_{i\in I}$, where $|J| \le \aleph_0$, such that $\mathbb{H}(b_j) \ge \inf_{\mathfrak{M}}\{\mathbb{H}(a_\tau)\}_{\tau\in J}$. Let $|J| = \aleph_0$. Since the system $\{G_i\}_{i\in I}$ satisfies the condition of finiteness, there exists a finite subset $\{a_{\tau_l}\}_{l=\overline{1,\,m}} \subset \{a_\tau\}_{\tau\in J}$ such that $\mathbb{H}(b_j) \ge \inf_{\mathfrak{M}}\{\mathbb{H}(a_{\tau_l})\}_{l=\overline{1,\,m}}$.

Let's demonstrate that there exists $\psi \in \mathrm{E}\,(G)$ such that $\psi(a) = b$. To do this, it is sufficient to see that for an arbitrary coordinate b_j of the element b there exists a homomorphism $\psi_j \in \mathrm{Hom}(G,\, G_j)$ such that $\psi_j(a) = b_j$. Then, by [46, Theorem 8.2], existence of the required homomorphism ψ follows. Let $\mathbb{H}(b_j) \ge \inf_{\mathfrak{M}}\{\mathbb{H}(a_{i_k})\}_{i_k\in I,\, k=\overline{1,\,n}}$, where $b_j \in G_j$, $a_{i_k} \in G_{i_k}$. Since the system $\{G_i\}_{i\in I}$ is fully transitive and stisfies the condition of monotonicity, the subsystem $\{G_j,\, G_{i_k}\}_{k=\overline{1,\,n}}$, where $G_j \ne G_{i_k}$ for all $k = \overline{1,\,n}$, is also fully transitive and satisfies the condition of monotonicity. Therefore, the group $G' = G_j \oplus \bigoplus\limits_{k=1}^{n} G_{i_k}$, as it follows from Theorem 1.3, is fully transitive. So there exists a homomorphism $\psi'_j \in \mathrm{Hom}(G',\, G_j)$ such that $(\psi'_j\rho')(a_{i_1} + \ldots + a_{i_n}) = b_j$, where $\rho' : \bigoplus\limits_{k=1}^{n} G_{i_k} \longrightarrow G'$ is an embedding, $\psi'_j = \pi'_j\psi''_j$, $\pi'_j : G' \longrightarrow G_j$ is a projection, and $\psi''_j \in \mathrm{E}\,(G')$, where $(\psi''_j\rho')(a_{i_1} + \ldots + a_{i_n}) = \rho_j(b_j)$, $\rho_j : G_j \longrightarrow G'$ is an embedding. Since G' is a direct summand of the group G, one can extend the homomorphism $\psi'_j\rho'$ to a homomorphism $\psi_j \in \mathrm{Hom}(G,\, G_j)$, mapping a to the element b_j. And if the group G_j coincide with one of the groups G_{i_k}, $k = \overline{1,\,n}$, similar reasoning for the subsystem $\{G_{i_k}\}_{k=\overline{1,\,n}}$, yield the homomorphism $\psi_j \in \mathrm{Hom}(G,\, G_j)$, which maps the element a to the element b_j.

The following example shows that there exists a fully transitive group $G = \prod\limits_{i\in I} G_i$ such that the system $\{G_i\}_{i\in I}$ does not satisfy the condition of finiteness.

Example 1.2. Let $A = \prod\limits_{i=0}^{\infty} A_i$, where A_0 is the group of integer p-adic numbers, and A_i $(i = \overline{1, \infty})$ are cyclic groups of order p^i (here p is a fixed prime number). Since each group A_i $(i = \overline{0, \infty})$ is algebraically compact, A is an algebraically compact group [46], and, as follows from Corollary 3.10, it is fully transitive. Consider the elements $a_i \in A_i$ $(i = \overline{0, \infty})$ such that $h_p(a_i) = 0$, then $\mathbb{H}(a_0) = \inf_{\mathfrak{M}}\{\mathbb{H}(a_i)\}_{i=\overline{1,\infty}}$. At the same time, there exist no finite subset of elements $\{a_{i_k}\}_{k=\overline{1,n}} \subset \{a_i\}_{i=\overline{1,\infty}}$ such that $\mathbb{H}(a_0) = \inf_{\mathfrak{M}}\{\mathbb{H}(a_{i_k})\}_{k=\overline{1,n}}$.

§2. Fully transitivity of torsion free groups

In this section we demonstrate that the condition of monotonicity for a system of pure rank 1 subgroups of an arbitrary torsion free group is equivalent to homogeneity. Then we characterize fully transitive separable torsion free groups and consider some problems connected with fully transitivity of direct product of generally separable, separable, and homogeneously decomposable torsion free groups.

Lemma 2.1. *A system of homogeneous torsion free groups $\{A_i\}_{i\in I}$ satisfies the condition of monotonicity if and only if $\pi(A_\alpha)\bigcap\pi(A_\beta) = \emptyset$ for all groups A_α, $A_\beta \in \{A_i\}_{i\in I}$ such that $t(A_\alpha) \neq t(A_\beta)$.*

Proof. Let the system of homogeneous torsion free groups $\{A_i\}_{i\in I}$ satisfy the condition of monotonicity and there exists groups A_α, $A_\beta \in \{A_i\}_{i\in I}$ such that $t(A_\alpha) \neq t(A_\beta)$ and $\pi(A_\alpha)\bigcap\pi(A_\beta) \neq \emptyset$. Let, for definiteness, $t(A_\alpha) \not\geq t(A_\beta)$ and p is a prime number such that $p \in \pi(A_\alpha)\bigcap\pi(A_\beta)$. Then, from the groups A_α and A_β, we respectively choose elements a_α and a_β such that $h_p(a_\beta) \leq h_p(a_\alpha)$. Then $\chi(a_\alpha) \geq \inf\{\chi(pa_\alpha), \chi(a_\beta)\}$, and $\chi(a_\alpha) < \chi(pa_\alpha)$. Since $h_p(a_\alpha) + 1 = h_p(pa_\alpha)$ and $\chi(a_\alpha) \not\geq \chi(a_\beta)$.

Since the system $\{A_i\}_{i\in I}$ satisfies the condition of monotonicity, there exist elements $a_{\alpha_1}, a_{\alpha_2}, \dots, a_{\alpha_n} \in A_\alpha$ such that $a_\alpha = a_{\alpha_1} + \dots + a_{\alpha_n}$, where $\chi(a_{\alpha_k}) \geq \chi(a_\beta)$ for any $k = \overline{1, n}$ or $\chi(a_{\alpha_k}) \geq \chi(pa_\alpha)$. Since $t(A_\alpha) \ngeq t(A_\beta)$, $\chi(a_{\alpha_k}) \ngeq \chi(a_\beta)$ for any $k = \overline{1, n}$. So $\chi(a_{\alpha_k}) \geq \chi(pa_\alpha)$ for each $k = \overline{1, n}$. Then $\chi(a_\alpha) \geq \inf\{\chi(a_{\alpha_k})\}_{k=\overline{1,n}} \geq \chi(pa_\alpha)$, what is impossible.

Conversely. Consider an arbitrary group $A_j \in \{A_i\}_{i\in I}$ and an arbitrary element $0 \neq a_j \in A_j$ such that $\chi(a_j) \geq \inf\{\chi(a_{i_1}), \dots, \chi(a_{i_s})\}$, where $a_{i_k} \in A_{i_k}$, $i_k \in I$, $k = \overline{1, s}$, $i_k \neq i_l$ for $k \neq l$, and $\chi(a_j) \ngeq \chi(a_{i_k})$ for any $k = \overline{1, s}$. Since the system $\{A_i\}_{i\in I}$ consists of reduced groups, there exists a prime number p such that $p \in \pi(A_j)$. Since $h_p(a_j) \geq \inf\{h_p(a_{i_k})\}_{k=\overline{1,s}}$, there exists an element $a_{i_\beta} \in \{a_{i_k}\}_{k=\overline{1,s}}$ such that $h_p(a_j) \geq h_p(a_{i_\beta})$. Then, as follows from the condition, $t(A_j) = t(A_{i_\beta})$. Since $h_q(a_j) = h_q(a_{i_\beta})$, is valid for almost all prime numbers q, let $q_1, \dots, q_n$ be prime numbers (different from p) for which $h_{q_\alpha}(a_j) \neq h_{q_\alpha}(a_{i_\beta})$. Then, for each prime number q_α ($\alpha = \overline{1, n}$), there exists an element $a_i^{(\alpha)} \in \{a_{i_k}\}_{k=\overline{1,s}}$ such that $h_{q_\alpha}(a_i^{(\alpha)}) \leq h_{q_\alpha}(a_j)$. By construction of the system $\{a_{i_\beta}, a_i^{(\alpha)}\}_{\alpha=\overline{1,n}}$, it follows that $t(a_j) = t(c)$ for any element c of this system, and $\chi(a_j) \geq \inf\{\chi(a_{i_\beta}), \chi(a_i^{(\alpha)})\}_{\alpha=\overline{1,n}}$. Since the system $\{A_j, A_{i_\beta}, A_i^{(\alpha)}\}_{\alpha=\overline{1,n}}$ consists of groups of the same type, it satisfies the condition of monotonicity [29, Lemma 2.2]. So there exist elements $a_{j_1}, \dots, a_{j_r} \in A_j$ such that $a_j = a_{j_1} + \dots + a_{j_r}$, and for each a_{j_l} ($l = \overline{1, r}$) there exists an element $d \in \{a_{i_\beta}, a_i^{(\alpha)}\}_{\alpha=\overline{1,n}}$ such that $\chi(a_{j_l}) \geq \chi(d)$. Then $\{a_{i_\beta}, a_i^{(\alpha)}\}_{\alpha=\overline{1,n}} \subseteq \{a_{i_k}\}_{k=\overline{1,s}}$.

Lemma 2.2. *If the groups G_i ($i \in I$) are homogeneous torsion free direct summands of an arbitrary fully transitive group G, the system $\{G_i\}_{i\in I}$ satisfies the condition of monotonicity.*

Proof. Let the condition of the lemma be satisfied. Then, by Lemma 2.1, we have to show that $\pi(A)\bigcap\pi(B) = \emptyset$ for all groups $A, B \in \{G_i\}_{i\in I}$ such that $t(A) \neq t(B)$. Let, for definiteness, $t(A) \ngeq t(B)$. If the groups A and B lay in the same decomposition of G, the statement of the Lemma follows from Theorem 1.3 and Lemma 2.1. Let the groups A and B lay in different decompositions of the group G, i.e., $A \oplus A_1 = G = B \oplus B_1$. Consider an arbitrary group $C \in \{A, B\}$ and an arbitrary element $0 \neq c \in C$ such that $\mathbb{H}(c) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a), \mathbb{H}(b)\}$, where $a \in A$, $b \in B$. Here $\mathbb{H}(c) \ngeq \mathbb{H}(a)$ and $\mathbb{H}(c) \ngeq \mathbb{H}(b)$. Let's show that $b \in A_1$. Indeed, since $t(A) \ngeq t(B)$ and $b \neq 0$,

$b \notin A$. Let $b = a' + a'_1$, where $a' \in A$, $a'_1 \in A_1$ (we consider the decomposition $G = A \oplus A_1$), then $\mathbb{H}(b) = \inf_{\mathfrak{M}}\{\mathbb{H}(a'), \mathbb{H}(a'_1)\} \leq \mathbb{H}(a')$. This contradicts to $t(A) \not\geq t(B)$. Therefore, $b \in A_1$, then $\mathbb{H}(a + b) \leq \mathbb{H}(c)$. Existence of $\varphi \in \mathrm{E}\,(G)$ such that $c = \varphi(a+b) = \varphi(a)+\varphi(b)$ follows from fully transitivity of the group G. Since $\mathbb{H}(\varphi(a)) \geq \mathbb{H}(a)$ and $\mathbb{H}(\varphi(b)) \geq \mathbb{H}(b)$, the system $\{A, B\}$ satisfies the condition of monotonicity and, by Lemma 2.1, it follows that $\pi(A) \bigcap \pi(B) = \emptyset$.

The following Lemma says that the condition of monotonicity for a system of pure rank 1 subgroups of a torsion free group G significantly influences upon its structure.

Lemma 2.3. *A reduced torsion free group G is homogeneous if and only if the system $\{A_i\}_{i \in I}$ of all its pure rank 1 subgroups satisfies the condition of monotonicity.*

Proof. Necessity follows from Lemma 2.1. Let's prove sufficiency. Suppose the contrary, i.e., let G be a non-homogeneous group. Then there exist non-zero elements a, $b \in G$ such that $t(a) \neq t(b)$. As follows from Lemma 2.1, $\pi(< a >_*) \cap \pi(< b >_*) = \emptyset$. Therefore, $t(a + b) = t(a) \cap t(b)$, what implies

$$t(a + b) < t(a) \quad \pi(< a + b >_*) \cap \pi(< a >_*) = \pi(< a >_*) \neq \emptyset.$$

Then, by Lemma 2.1, the system $\{< a + b >_*, < a >_*\}$ does not satisfy the condition of monotonicity what contradicts to our supposition and to the note before Theorem 1.3.

The following Theorem characterizes fully transitive separable torsion free groups.

Theorem 2.4. *For a separable torsion free group G, the following are equivalent:*
1) G is a homogeneously decomposable group, and $\pi(G_i) \bigcap \pi(G_j) = \emptyset$ for all homogeneous direct summands G_i, G_j from G such that $t(G_i) \neq t(G_j)$;
2) $\pi(A) \bigcap \pi(B) = \emptyset$ for all non-isomorphic rank 1 direct summands A and B of G;
3) the system $\{A_i\}_{i \in I}$ of all rank 1 direct summands of the group G satisfies

the condition of monotonicity;
4) the group G is fully transitive.

Proof. The implication 1) $\Rightarrow$ 4) follows from [29, Corollary 2.13]; using Lemma 2.2, we obtain 3) from 4); equivalence of 2) and 3) is established in Lemma 2.1. Let's show that 1) follows from 2). Let $\tau(G)$ denote the set of different types of all rank 1 direct summands in G. Then, for each $\tau \in \tau(G)$, consider the subgroups $G^{(\tau)}$ purely generated by all rank 1 direct summands of type τ. Let's show that $G^{(\tau)}$ is a homogeneous group of type τ, i.e., that an arbitrary element $g \in G^{(\tau)}$ is of type τ in $G^{(\tau)}$. Since the element g is a solution of the equation $nx = a_{j_1} + \ldots + a_{j_k}$ in G, where $a_{j_l} \in A_{j_l}$, $t(A_{j_l}) = \tau$ and $A_{j_l} \in \{A_i\}_{i\in I}$ for any $l = \overline{1, k}$ (here $\{A_i\}_{i\in I}$ is the system of all rank 1 direct summands of the group G), we can prove that

$$t(a_{j_1} + \ldots + a_{j_k}) = t(a_{j_1}) \cap \ldots \cap t(a_{j_k}). \tag{$*$}$$

by induction on k.

Let $k = 2$. Then the group G can be represented as $G = A_{j_1} \oplus H$. If $a_{j_2} \in H$, $t(a_{j_1} + a_{j_2}) = t(a_{j_1}) \cap t(a_{j_2})$. If $a_{j_2} \in A_{j_1}$, $a_{j_1} + a_{j_2} \in A_{j_1}$ and $t(a_{j_1} + a_{j_2}) = t(a_{j_1}) \cap t(a_{j_2})$. If $a_{j_2} = a'_{j_1} + h$, where $a'_{j_1} \in A_{j_1}$, $h \in H$, $a_{j_1} + a_{j_2} = a_{j_1} + a'_{j_1} + h$ and $t(a_{j_1} + a_{j_2}) = t(a_{j_1} + a'_{j_1} + h) = t(a_{j_1} + a'_{j_1}) \cap t(h) = t(a_{j_1}) \cap t(a'_{j_1}) \cap t(h)$. Since $t(a_{j_2}) = t(a'_{j_1}) \cap t(h)$, $t(a_{j_1}) \cap t(a_{j_2}) = t(a_{j_1} + a_{j_2})$. Let the equality $(*)$ be valid for $k = m - 1$ $(m \geq 3)$. Let's demonstrate its validity for $k = m$. Using the induction hypotheses, we obtain that $t(a_{j_1} + \ldots + a_{j_k}) = t(a_{j_1} + \ldots + a_{j_{k-1}}) \cap t(a_{j_k}) = t(a_{j_1}) \cap \ldots \cap t(a_{j_{k-1}}) \cap t(a_{j_k})$. Note also that for all τ_1, $\tau_2 \in \tau(G)$ it follows that $t(G^{(\tau_1)}) \neq t(G^{(\tau_2)})$, and since $t(G^{(\tau_1)}) = t(A^{(\tau_1)})$, $t(G^{(\tau_2)}) = t(A^{(\tau_2)})$ for some $A^{(\tau_1)}$, $A^{(\tau_2)} \in \{A_i\}_{i\in I}$, and the system $\{A_i\}_{i\in I}$ satisfies the condition 2), we have $\pi(G^{(\tau_1)}) \cap \pi(G^{(\tau_2)}) = \emptyset$. Let's show that $G = \bigoplus_{\tau\in\tau(G)} G^{(\tau)}$. Let $G^{(\alpha)}$, $G^{(\tau_1)}, \ldots, G^{(\tau_k)} \in \{G^{(\tau)}\}_{\tau\in\tau(G)}$, where $G^{(\tau_i)} \neq G^{(\tau_j)}$, $i \neq j$ for all i, $j = \overline{1, k}$ and $G^{(\alpha)} \neq G^{(\tau_i)}$ for any $i = \overline{1, k}$. Let's assume that there exists $0 \neq b \in G^{(\alpha)} \cap (G^{(\tau_1)} + \ldots + G^{(\tau_k)})$. Since G is a reduced group, there exists a prime number p such that $h_p(b) \neq \infty$ in the group $G^{(\alpha)}$, and, consequently, in the group G. On the other hand, since $\pi(G^{(1)}) \cap \pi(G^{(2)}) = \emptyset$ for all $G^{(1)}$, $G^{(2)} \in \{G^{(\tau)}\}_{\tau\in\tau(G)}$, $G^{(1)} \neq G^{(2)}$, we have $t(G^{(\tau_1)} + \ldots + G^{(\tau_k)}) = t(G^{(\tau_1)}) \cap \ldots \cap t(G^{(\tau_k)})$. And since $\pi(G^{(\alpha)}) \cap \pi(G^{i}) = \emptyset$ for any $i = \overline{1, k}$, we have $\pi(G^{(\alpha)}) \cap \pi(G^{(\tau_1)} + \ldots + G^{(\tau_k)}) = \emptyset$. Therefore, $h_p(b) = \infty$ in the group $G^{(\tau_1)} + \ldots + G^{(\tau_k)}$, and in the group G, what contradicts uniqueness of height of an element in a group. Let's show that the group G is generated by the set $\{G^{(\tau)}\}_{\tau\in\tau(G)}$. Indeed, let's assume that there exists an element $g \in G$ such that $g \notin \bigoplus\limits_{\tau\in\tau(G)} G^{(\tau)}$. Since the group G is separable, we obtain that the element g can be embedded into a completely decomposable direct summand $F = F_1 + \ldots + F_t$ of the group G. For each $l = \overline{1, t}$ there exists $\tau_l \in \tau(G)$ such that $F_l \subseteq G^{(\tau_l)}$. So the element $g \in \bigoplus\limits_{l=1}^{t} G^{(\tau_l)}$, what contradicts our assumption.

Definition 2.1. *We say that a system of torsion free groups* $\{G_i\}_{i\in I}$ *is endotransitive if* $a \in G_i$, $b \in G_j$ *and* $\chi(a) = \chi(b)$, *imply existence of* $\varphi \in \mathrm{Hom}(G_i, G_j)$ *with the property* $\varphi(a) = b$.

If the system $\{G_i\}_{i\in I}$ consists of a single group, we come to the concept of endotransitivity of a group introduced in [35].

The following simple proposition shows that the concepts of fully transitivity and endotransitivity coincide for a system consisting of homogeneous groups of the same type.

Lemma 2.5. *An arbitrary system of homogeneous groups* $\{G_i\}_{i\in I}$ *of the same type is fully transitive if and only if it is endotransitive.*

Proof. Necessity is evident. Let's prove sufficiency. Consider arbitrary groups $G_j, G_t \in \{G_i\}_{i\in I}$ and arbitrary elements $a \in G_j$, $b \in G_t$ such that $\chi(a) \le \chi(b)$. If $\chi(a) = \chi(b)$, there exists a homomorphism $\psi \in \mathrm{Hom}(G_j, G_t)$ such that $\psi(a) = b$. Now let $\chi(a) < \chi(b)$. Since the characteristics $\chi(a)$ and $\chi(b)$ are equivalent, there exists a natural number n such that $\chi(na) = \chi(b)$. So there is a homomorphism $\varphi \in \mathrm{Hom}(G_j, G_t)$ such that $\varphi(na) = b$. Therefore, the homomorphism $\psi = n\varphi$ maps a into the element b.

We recall that a torsion free group G is said to be *generally separable* [29] if each finite subset of elements from G is contained in a certain homogeneously decomposable direct summand of the group G.

Theorem 2.6. *A group* $A = \prod\limits_{i\in I} A_i$ *where* A_i $(i \in I)$ *are generally separable groups is fully transitive if and only if for all homogeneous direct summands* B *and* C *from* A *the following conditions*

1) the system $\{B,\, C\}$ is endotransitive;
2) if $t(B) \neq t(C)$, then $\pi(B) \cap \pi(C) = \emptyset$
are fulfilled.

Proof. Necessity. The condition 2) follows from Lemmas 2.1 and 2.2. Let's demonstrate validity of the condition 1). As follows from Theorem 1.3 and Lemma 2.5, the system of groups $\{B,\, C\}$ satisfying the condition of the Theorem is endotransitive if these groups lay in the same decomposition of the group A and if $t(B) \neq t(C)$, because in this case $\mathrm{Hom}(B,\, C) = \mathrm{Hom}(C,\, B) = 0$. Let $t(B) = t(C)$ and the groups $B,\, C$ lay in different decompositions of the group A. Consider arbitrary elements $b \in B$, $c \in C$ such that $\chi(b) = \chi(c)$, then $\chi(\rho_1(b)) = \chi(\rho_2(c))$, where $\rho_1 : B \longrightarrow A$, $\rho_2 : C \longrightarrow A$ are embeddings. Fully transitivity of the group A implies existence of $\varphi \in \mathrm{E}\,(A)$ such that $\varphi(\rho_1(b)) = \rho_2(c)$. Then there exists $\psi \in \mathrm{Hom}(B,\, C)$ such that $\psi(b) = c$, where $\psi = \pi\varphi\rho_1$ and $\pi : A \longrightarrow C$ is a projection.

Sufficiency. Let $a,\, b \in A$, $\chi(a) \leq \chi(b)$ and $a = (\ldots,\, a_i,\, \ldots)$, $b = (\ldots,\, b_i,\, \ldots)$, where $a_i,\, b_i \in A_i$ for any $i \in I$. Since A_i is a generally separable group, one can embed b_i in a homogeneously decomposable direct summand. Let H_i be one of minimal (with respect to inclusion) summands of such kind. We have $b_i = b_{i_1} + \ldots + b_{i_m}$, where $H_i = H_{i_1} + \ldots + H_{i_m}$ is decomposition of H_i into a direct summand of homogeneous groups (where m depends on the index i and on the choice of H_i). We say that such a decomposition of the element b_i is homogeneous. Since $(n = \overline{1,\, m})$ $\chi(b_{i_n}) \geq \chi(a)$ for each summand b_{i_n}, it is sufficient to show that there exists $\psi_{i_n} \in \mathrm{Hom}(A,\, H_{i_n})$ such that $\psi_{i_n}(a) = b_{i_n}$. Then $\sum\limits_{n=1}^{m} \psi_{i_n}(a) = b_i$, where $\sum\limits_{n=1}^{m} \psi_{i_n} \in \mathrm{Hom}(A,\, A_i)$. Then, as follows from [46, Theorem 8.2], there exists $\psi \in \mathrm{E}\,(A)$ such that $\psi(a) = b$. Since the condition 2) of this Theorem is satified and $\chi(b_{i_n}) \geq \chi(a)$, one can find a system of elements $\{a_{i_l}\}_{l=\overline{1,\, d}}$ such that $a_i = a_{i_1} + \ldots + a_{i_d}$ and $t(a_{i_l}) = t(b_{i_n})$ for any $l = \overline{1,\, d}$, where $\chi(b_{i_n}) \geq \inf\{\chi(a_{i_l})\}_{l=\overline{1,\, d}}$. The homogeneous system $\{H_{i_l},\, H_{i_n}\}$ is fully transitive by virtue of condition 1) and Lemma 2.5. So, if there exists $l = \overline{1,\, d}$ such that $\chi(a_{i_l}) \leq \chi(b_{i_n})$, existence of $\varphi_{i_l} \in \mathrm{Hom}(H_{i_l},\, H_{i_n})$ such that $\varphi_{i_l}(a_{i_l}) = b_{i_n}$ follows from fully transitivity of this system. Then there exists $\psi_{i_n} \in \mathrm{Hom}(A,\, H_{i_n})$ mapping the element a into the element b_{i_n}, where $\psi_{i_n} = \varphi_{i_l}\pi_{i_l}\pi$ and $\pi : A \longrightarrow A_i$, $\pi_{i_l} : A_i \longrightarrow H_{i_l}$ are projections. If there are no such an element a_{i_l} $(l = \overline{1,\, d})$, the system $\{H_{i_l},\, H_{i_n}\}_{l=\overline{1,\, d}}$, consisting of homogeneous groups of the same type, satisfies the condition of monotonicity (this follows from Lemma 2.1 and condition

2) of the Theorem). Then there exist elements $b_{i_n}^{(1)}, \dots, b_{i_n}^{(\gamma)} \in H_{i_n}$ such that $b_{i_n} = b_{i_n}^{(1)}, \dots, b_{i_n}^{(\gamma)}$, and for each $b_{i_n}^{(t)}$ $(t = \overline{1, \gamma})$, there exists an element $a_{i_l}^{(t)} \in \{a_{i_l}\}_{l=\overline{1,d}}$, for which $\chi(b_{i_n}^{(t)}) \geq \chi(a_{i_l}^{(t)})$. Since the system $\{H_{i_l}^{(t)}, H_{i_n}\}$ is fully transitive for each $t = \overline{1, \gamma}$, there exist $\varphi_{i_t} \in \mathrm{Hom}(H_{i_l}^{(t)}, H_{i_n})$ $(t = \overline{1, \gamma})$, mapping the elements $a_{i_l}^{(t)}$ into the elements $b_{i_n}^{(t)}$ respectively (here $H_{i_l}^{(t)} \in \{H_{i_l}\}_{l=\overline{1,d}}$ for each $t = \overline{1, \gamma}$). Then $\sum\limits_{t=1}^{\gamma} \varphi_{i_t}(a_{i_l}^{(t)}) = b_{i_n}$. Therefore, there exists a homomorphism $\psi_{i_t} = \sum\limits_{t=1}^{\gamma} \varphi_{i_t}\pi_{i_t}\pi_i$ such that $\psi_{i_t} a = b_{i_n}$, where $\pi_{i_t} : A_i \longrightarrow H_{i_t}$ $\pi_i : A \longrightarrow A_i$ are projections.

Corollary 2.7. *A group* $A = \prod\limits_{i \in I} A_i$, *where* A_i $(i \in I)$ *are separable groups, is fully transitive if and only if* $\pi(B) \bigcap \pi(C) = \emptyset$ follows for all non-isomorphic rank 1 direct summands B and C of the group A.

Corollary 2.8.[35] *A vector group* A *is fully transitive if and only if* $\pi(B) \bigcap \pi(C) = \emptyset$ follows for all non-isomorphic rank 1 direct summands B and C of the group A.

Corollary 2.9.[29] *A completely decomposable group* A *is fully transitive if and only if* $\pi(B) \bigcap \pi(C) = \emptyset$ follows for all non-isomorphic rank 1 direct summands B and C of the group A.

§3. Fully transitivity of direct products and direct sums of s-generally slender groups

In this section we prove a criterion for fully transitivity of direct products of **s**-generally slender groups (Theorem 3.1). By use of this, we obtain some equivalent conditions for fully transitivity of direct products of generally slender groups, separable groups, countable groups, slender torsion free groups, torsion groups (Corollaries 3.2 – 3.4). Besides, we demonstrate that any algebraically compact group is fully transitive.

As it was mentioned in §1, the conditions of fully transitivity, monotonicity, and finiteness of an arbitrary system of groups $\{G_i\}_{i \in I}$ are only sufficient for fully transitivity of the group $\prod\limits_{i \in I} G_i$. In this section we introduce a class of groups for which these conditions are also necessary for fully transitivity of their direct product if we impose them on an arbitrary system of groups from this class.

The following concept was introduced by S. V. Rychkov in [45].

Definition 3.1. [45] *An abelian group A is said to be generally slender if it does not contain non-bounded coperiodic subgroups and subgroups isomorphic to the group $\prod\limits_{\aleph_0}\mathbb{Z}$.*

Slender, countable reduced, and torsion reduced groups are examples of generally slender groups [45].

Definition 3.2. *We say that a group G is* **s***-generally slender if each element $g \in G$ such that $o(g) = \infty$ can be embedded into a generally slender direct summand of the group G.*

The class of **s**-generally slender groups is wider than the class of generally slender groups. In particular, it contains all separable reduced abelian groups including $\prod\limits_{\aleph_0}\mathbb{Z}$. Indeed, let A be a separable reduced group. We can show that it is **s**-generally slender. Let $a \in A$, $o(g) = \infty$. Then, since the group A is separable, the element a can be embedded into a completely decomposable direct summand B of the group A, where $B = \mathop{\oplus}\limits_{i=1}^{n} B_i$, B_i $(i = \overline{1, n})$ are rank 1 groups. Every B_i is a generally slender group. So, as it follws from [45, Proposition 3], B is a generally slender group.

Now we consider one of the main results of this section.

Theorem 3.1. *A group $G = \prod\limits_{i\in I} G_i$, where G_i $(i \in I)$ are* **s***-generally slender groups, is fully transitive if and only if the following conditions are satisfied:*
1) $\{G_i\}_{i\in I}$ is a fully transitive system of groups;
2) the system of groups $\{G_i\}_{i\in I}$ satisfies the condition of monotonicity;
3) the system of groups $\{G_i\}_{i\in I}$ satisfies the condition of finiteness.

Proof. Sufficiency follows from Proposition 1.5. Necessity. Fulfilment of the conditions 1) – 2) follows from Proposition 1.4. Let's demonstrate validity of the condition 3). Consider an arbitrary group $G_j \in \{G_i\}_{i\in I}$ and an arbitrary element $g_j \in G_j$ such that $o(g_j) = \infty$ and $\mathbb{H}(g_j) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(g_\tau)\}_{\tau\in J}$, $J \subseteq I$, and $|J| = \aleph_0$. Then $G = \prod\limits_{\tau\in J} G_\tau \oplus \prod\limits_{i\in I\setminus J} G_i$. Let $G^{(1)} = \prod\limits_{\tau\in J} G_\tau$ and $G^{(2)} = \prod\limits_{i\in I\setminus J} G_i$. Since G is a fully transitive group, the system $\{G^{(1)}, G^{(2)}\}$ is fully transitive. Two cases are possible: $j \in I \setminus J$ or $j \in J$. Let $j \in I \setminus J$ and $g^{(1)} = (\ldots, g_\tau, \ldots) \in G^{(1)}$, $g^{(2)} = (0, \ldots, 0, g_j, 0, \ldots) \in G^{(2)}$. Since $\mathbb{H}(g^{(2)}) \geq \mathbb{H}(g^{(1)})$, there exists $\varphi \in \mathrm{Hom}(G^{(1)}, G^{(2)})$ such that $\varphi(g^{(1)}) = g^{(2)}$. Then one can find a homomorphism $\psi \in \mathrm{Hom}(G^{(1)}, B_j)$ such that $\psi(g^{(1)}) = g_j$, where B_j is a generally slender direct summand of the group G_j, $\psi = \pi\pi_j\varphi$; $\pi_j : G^{(2)} \longrightarrow G_j$, $\pi : G_j \longrightarrow B_j$ are projections. Since B_j is a generally slender group, there exists, as follows from [45, Proposition 1], a natural number k such that $\psi(\prod\limits_{\tau=k}^{\infty} G_\tau)$ is a bounded group. The group $G^{(1)}$ can be represented in the form $G^{(1)} = G_1 \oplus \ldots \oplus G_{k-1} \oplus \prod\limits_{\tau=k}^{\infty} G_\tau$, then the element $g^{(1)}$ can be written as $g^{(1)} = g_1 + \ldots + g_{k-1} + \overline{g}$, where $g_t \in G_t$ ($t = \overline{1, k-1}$), $G_t \in \{G_\tau\}_{\tau=\overline{1,\infty}}$ and $\overline{g} \in \prod\limits_{\tau=k}^{\infty} G_\tau$. Then the element $g_j = \psi(g^{(1)}) = \psi(g_1 + \ldots + g_{k-1}) + \psi(\overline{g}) = g_{j_1} + g_{j_2}$, $g_{j_1} = \psi(g_1 + \ldots + g_{k-1})$ and $g_{j_2} = \psi(\overline{g})$. Since $g_{j_2} \in \psi(\prod\limits_{\tau=k}^{\infty} G_\tau)$, $o(g_{j_2}) < \infty$. From $\mathbb{H}(g_{j_2}) \geq \mathbb{H}(\overline{g})$ and $o(g_{j_2}) < \infty$ follows existence of a finite coordinate system $\{g_{\tau_l}\}_{\tau_l=\overline{k,\infty};\, l=\overline{1,m}}$ for the element $\overline{g}$ such that $\inf_{\mathfrak{M}}\{\mathbb{H}(g_{\tau_l})\}_{\tau_l=\overline{k,\infty};\, l=\overline{1,m}} \leq \mathbb{H}(g_{j_2})$. Then

$$\begin{aligned}\mathbb{H}(g_j) &\geq \inf_{\mathfrak{M}}\{\mathbb{H}(g_{j_1}), \mathbb{H}(g_{j_2})\}\\ &\geq \inf\{\inf_{\mathfrak{M}}\{\mathbb{H}(g_t)\}_{t=\overline{1,k-1}}, \inf_{\mathfrak{M}}\{\mathbb{H}(g_{\tau_l})\}_{\tau_l=\overline{k,\infty};\, l=\overline{1,m}}\}\\ &= \inf_{\mathfrak{M}}\{\mathbb{H}(g_{j_1}), \ldots, \mathbb{H}(g_{k-1}), \mathbb{H}(g_{\tau_1}), \ldots, \mathbb{H}(g_{\tau_m})\}.\end{aligned}$$

The case $j \in J$ can be proved similarly.

The following corollaries can be obtained immediately from this Theorem.

Corollary 3.2. *A group $G = \prod\limits_{i\in I} G_i$, where G_i ($i \in I$) are generally slender groups, is fully transitive if and only if the conditions 1) – 3) of the* *previous Theorem are satisfied for the system* $\{G_i\}_{i\in I}$.

Corollary 3.3. *If each group* G_i $(i \in I)$ *satisfies at least one of the following conditions:*
1) G_i *is a separable group;*
2) G_i *is a countable group;*
3) G_i *is a slender group;*
4) G_i *is a torsion group,*
the group $G = \prod\limits_{i\in I} G_i$ *is fully transitive if and only if the conditions 1) – 3) of Theorem 3.1 are satisfied.*

Corollary 3.4. *A group* $G = \prod\limits_{i\in I} G_i$*, where* G_i $(i \in I)$ *are torsion groups, is fully transitive if and only if the conditions 1), 2) of Theorem 3.1 are satisfied.*

Further, we show that the criterion for fully transitivity of the direct product of torsion groups can be simplified. First, we prove the following working lemma.

Lemma 3.5. *A system of torsion groups* $\{G_i\}_{i\in I}$ *satisfies the condition of monotonicity for height matrices if and only if for each prime number* p *the system* $\{G_{ip}\}_{i\in I}$ *(where* G_{ip} *are* p*-primary components of the group* G_i*) satisfies the condition of monotonicity.*

Proof. Necessity directly follows from the definition of monotonicity of a system. Let's prove sufficiency. Consider an arbitrary group $G_j \in \{G_i\}_{i\in I}$ and an arbitrary element $g_j \in G_j$ such that $\mathbb{H}(g_j) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(g_{i_k})\}_{i_k\in I,\, k=\overline{1,\, n}}$. Here $g_{i_l} \neq g_{i_t}$ if $i_l \neq i_t$. Let $\mathbb{H}(g_j) \ngeq \mathbb{H}(g_{i_k})$ for any $k = \overline{1,\, n}$. Let's represent the elements g_j and $\{g_{i_k}\}_{k=\overline{1,\, n}}$ as a sum of elements whose orders are powers of different prime numbers:

$$g_j = g_{j_1} + \ldots + g_{j_r}\ g_{i_k} = g_{i_k}^{(1)} + \ldots + g_{i_k}^{(m)}\ (k = \overline{1,\, n}).$$

Then

$$\mathbb{H}(g_{j_\alpha}) \geq \mathbb{H}(g_j) \geq \inf{}_{\mathfrak{M}}\{\mathbb{H}(g_{i_k})\}_{i_k\in I,\, k=\overline{1,\, n}} = \inf{}_{\mathfrak{M}}\{\mathbb{H}(g_{i_k}^{(\beta)})\}_{i_k\in I,\, k=\overline{1,\, n},\, \beta=\overline{1,\, m}}$$

for each $\alpha = \overline{1,\, r}$. Consider the elements g_{j_α} $(\alpha = \overline{1,\, r})$ such that $\mathbb{H}(g_{j_\alpha}) \ngeq \mathbb{H}(g_{i_k}^{(\beta)})$ for all $\beta = \overline{1,\, m}$ and $k = \overline{1,\, n}$ (if there are no such elements, the condition of monotonicity is satisfied trivially). Then, for each element g_{j_α}, one can find elements

$$\{a_\tau^{(\alpha)}\}_{\tau=\overline{1,t}} \subseteq \{g_{i_k}^{(\beta)}\}_{i_k\in I,\, k=\overline{1,n},\, \beta=\overline{1,m}}$$

whose orders are powers of the prime numbers equal to order of the element g_{j_α}. Here $\mathbb{H}(g_{j_\alpha}) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a_\tau^{(\alpha)})\}_{\tau=\overline{1,t}}$. Since the system $\{G_{ip}\}_{i\in I}$ satisfies the condition of monotonicity for each prime number p, there exist elements $b_{j_1}^{(\alpha)}, \ldots, b_{j_\gamma}^{(\alpha)}$ such that $b_{j_1}^{(\alpha)} + \ldots + b_{j_\gamma}^{(\alpha)} = g_{j_\alpha}$, and for each element $b_{j_\delta}^{(\alpha)}$ ($\delta = \overline{1,\,\gamma}$) there exists $a_\tau^{(\alpha)}$ ($\tau = \overline{1,\,t}$) such that $\mathbb{H}(b_{j_\delta}^{(\alpha)}) \geq \mathbb{H}(a_\tau^{(\alpha)})$. Then $\mathbb{H}(b_{j_\delta}^{(\alpha)}) \geq \mathbb{H}(g_{i_k})$ for a certain $k = \overline{1,\,n}$, because $a_\tau^{(\alpha)}$ is one of the summands in the decomposition of the element g_{i_k}. Further, replacing the elements g_{j_α} by their decomposition $b_{j_1}^{(\alpha)} + \ldots + b_{j_\gamma}^{(\alpha)}$, we obtain a new decomposition for the element g_j, and for each element c_{j_σ} from this decomposition one can find an element g_{i_k} ($k = \overline{1,\,n}$) such that $\mathbb{H}(c_{j_\sigma}) \geq \mathbb{H}(g_{i_k})$.

The following proposition is of some interest.

Proposition 3.6. *Any system of fully transitive torsion groups satisfies the condition of monotonicity.*

Proof follows from Lemma 3.5, [10, Proposition 1] and Theorem 1.3.

The following corollary follows from the Proposition 3.6, Corollary 3.4, and Theorem 1.3.

Corollary 3.7. *For torsion groups G_i $(i \in I)$, the following conditions are equivalent:*
1) $\prod\limits_{i\in I} G_i$ is a fully transitive group;
2) $\bigoplus\limits_{i\in I} G_i$ is a fully transitive group;
3) $\{G_i\}_{i\in I}$ is a fully transitive system of groups.

The following result shows that if a torsion group G has a separable direct summand A, the question about fully transitivity of the whole group can be reduced to the question about fully transitivity of the additional to A direct summand.

Proposition 3.8. *A torsion group $G = A \oplus B$ where A is a separable group is fully transitive if and only if the group B is fully transitive.*

Proof. Applying Theorem 1.3, we obtain necessity. Let's prove sufficiency. By virtue of Corollary 3.7, it remains to show that that the system $\{A, B\}$ is fully transitive or that for any prime number p the system $\{A_p, B_p\}$ (where A_p, B_p are p-components of the groups A and B respectively) is fully transitive. The groups A_p, B_p are fully transitive as direct summands of fully transitive groups. So, let's show that for all elements $a \in A_p$ and $b \in B_p$ such that $H_p(a) \leq H_p(b)$ ($H_p(b) \leq H_p(a)$) there exists $\varphi \in \mathrm{Hom}(A, B)$ ($\psi \in \mathrm{Hom}(B, A)$) such that $\varphi(a) = b$ ($\psi(b) = a$). Suppose that the subgroup $< b >$ does not contain elements of infinite p-height in B_p. Then the element b can be embedded in a finite direct summand B'_p of the group B_p [46, Corollary 27.9]. Since B'_p is a separable group, $A_p \oplus B'_p$ is separable and, consequently, fully transitive. Then, as follows from Corollary 3.7, there exists $\varphi \in \mathrm{Hom}(A_p, B'_p)$, mapping the element a into b, if $H_p(a) \leq H_p(b)$. On the other hand, there exists $\psi' \in \mathrm{Hom}(B'_p, A_p)$ such that $\psi'(b) = a$ if $H_p(b) \leq H_p(a)$. But then there exists $\psi \in \mathrm{Hom}(B_p, A_p)$ such that $\psi(b) = a$, where $\psi = \psi'\pi$ and $\pi : B_p \longrightarrow B'_p$ is a projection. Let the subgroup $< b >$ contain elements of infinite height. Then $H_p(a) < H_p(b)$ is a unique possible comparison (the heights of elements are taken in A_p and B_p respectively). Let δ_n be the least infinite ordinal in $H_p(b) = (\delta_0, \dots, \delta_k, \infty, \dots)$. Let $o(a) = p^t$ and $h_p(p^{t-1}a) = s$. If $n = 0$, the group B_p contains cyclic summands of arbitrarily high order because it is non-bounded. Let F_p be such a summand of order not less than p^{t+s}, i.e., let $|F_p| = p^{t+s+r}$, where $r \geq 0$ and f is the generator of the group F_p. Then $H_p(a) \leq H_p(p^{s+r}f)$ and, as follows from the previous case, there exists $\varphi \in \mathrm{Hom}(A_p, B_p)$, mapping the element a into $p^{s+r}f$. Since B_p is a fully transitive group and $o(p^{s+r}f) \geq o(b)$, one can find $\psi \in \mathrm{E}\,(B_p)$ such that $\psi(p^{s+r}f) = b$. Then $\psi\varphi \in \mathrm{Hom}(A_p, B_p)$ maps the element a into the element b. Let $n \neq 0$, then δ_ith Ulm-Kaplansky invariants are different from zero for all the ordinals δ_i ($0 \leq i < n$) followed by jumps [47, Lemma 65.3]. Therefore, the increasing surface $(\delta_0, \dots, \delta_{n-1}, \infty, \dots)$ is the characteristic function of a certain element $b'_p \in B_p$. Since the group $< b'_p >$ does not contain elements of infinite height, the element b'_p can be embedded in a finite direct summand B'_p of the group B_p. Let $B_p = B'_p \oplus B''_p$. Since B''_p is a non-bounded group, it contains cyclic direct summands of arbitrarily high. So, B''_p contains a cyclic direct summand C_p of order not less than p^{t+s}, i.e., $|C_p| = p^{t+s+r}$ ($r \geq 0$) and $c \in C_p$ is its generator.

Then $H_p(a) < H_p(b' + p^{s+r}c) < H_p(b)$. By virtue of Corollary 3.7, since $A_p \oplus B'_p \oplus C_p$ is a separable direct summand of the group G, there exists $\varphi \in \mathrm{Hom}(A_p, B'_p \oplus C_p)$ such that $\varphi(a) = b' + p^{s+r}c$. At the same time, fully transitivity of the group B_p implies existence of $\psi \in \mathrm{E}(B_p)$ such that $(\psi\varphi)(a) = b$, where $\psi\varphi \in \mathrm{Hom}(A_p, B_p)$.

Some examples of fully transitive (not obligatorily torsion) groups are presented by the following Corollary, where a separable (totally projective) torsion group means a torsion group for which each p-component is a separable (totally projective) group.

Corollary 3.9. *If each of the reduced torsion groups $G_i\,(i \in I)$ satisfies at least one of the following conditions:*
1) G_i is a separable group;
2) G_i is a totally projective group,
then $\prod\limits_{i\in I} G_i$ and $\bigoplus\limits_{i\in I} G_i$ are fully transitive groups.

Proof. By Lemma 3.5, one can consider that each group $G_i\,(i \in I)$ is a p-group. Let's show that $\bigoplus\limits_{i\in I} G_i$ is a fully transitive group. Let $I_1 = \{i \in I \mid G_i \text{ is a separable group}\}$, then $\bigoplus\limits_{i\in I} G_i = \bigoplus\limits_{i\in I_1} G_i \oplus \bigoplus\limits_{i\in I\setminus I_1} G_i$. Note that $\bigoplus\limits_{i\in I_1} G_i$ is a separable group ($\bigoplus\limits_{i\in I\setminus I_1} G_i$ is a totally projective group) as a direct sum of separable (totally projective groups) [47, p.122, Theorem 83.5]. Since each totally projective p-group is fully transitive [14], $\bigoplus\limits_{i\in I\setminus I_1} G_i$ is a fully transitive group. Using Proposition 3.8, we obtain that the group $\bigoplus\limits_{i\in I} G_i$ is fully transitive. Corollary 3.7 makes it possible to say that $\prod\limits_{i\in I} G_i$ is a fully transitive group.

Corollary 3.10. *An algebraically compact group is fully transitive.*

Proof follows from Theorem 1.3, Corollary 3.8, and [46, Corollary 38.2].

§4. Fully transitivity of direct products of abelian groups whose systems of factors satisfy the condition of finiteness

In this section we obtain simpler criteria for fully transitivity of split mixed groups, fully transitivity of direct products of separable groups. Influence of fully transitivity on split direct products of **s**-generally slender groups and separability of a direct product of groups is studied.

Let's prove the following simple statement.

Lemma 4.1. *If a group G contains an element a of order p^n, $n \in \mathbb{N}$ and generalized p-height greater or equal m (where m is a non-negative integer), it has a cyclic direct summand of order not less than p^{n+m}.*

Proof. Let the group G contain an element a satisfying the condition of the Lemma. Let $T_p(G)$ be the p-component of the torsion subgroup of the group G. As follows from [46, p.142], if $T_p(G)$ is a non-bounded group, it contains a cyclic direct summand of arbitrarily high order. This summand, being a bounded pure subgroup of the group G, is a direct summand in it. If $T_p(G)$ is a bonded group, let $h_p^*(a) = k \geq m$. Then, as follows from [41], the element $p^{n-1}a$ can be embedded into a cyclic direct summand B of order $p^{n+k} \geq p^{n+m}$.

Lemma 4.2. *Let A be a torsion free group and $\prod\limits_{i\in I} G_i$ be a fully transitive group, where $G_i\,(i \in I)$ are torsion groups. Then for all elements $a \in A$ and $g \in G$ such that $\mathbb{H}(a) \leq \mathbb{H}(g)$ there exixts $\varphi \in \mathrm{Hom}(A,\, G)$ mapping a into g.*

Proof. Let $a \in A$ and $g \in G$ satisfy the condition of the lemma and $g = (\dots, g_i, \dots)$, where $g_i \in G_i$. Since $\mathbb{H}(a) < \mathbb{H}(g_i)$ for any $i \in I$, it is sufficient to show that for any coordinate g_i, $(i \in I)$ there exists $\varphi_i \in \mathrm{Hom}(A,\, G_i)$ such that $\varphi_i(a) = g_i$. Then, as follows from [46, Theorem 8.2], there exists $\varphi \in \mathrm{Hom}(A,\, G)$ such that $\varphi(a) = g$. Let's consider an arbitrary coordinate $g_i \in G_i$ and denote it by b. Let the group G_i be denoted by B. Now we fix a set of prime numbers $p_1, \dots, p_k$ for which $h_{p_i}^*(b) \neq \infty$ and let $h_{p_i}(a) = n_i$ $(i = \overline{1,\, k})$. The element b can be represented in the form $b = b_1 + \ldots + b_k$, where $B_i\,(i = \overline{1,\, k})$ are p_i-components of the group B and $b_i \in B_i$. So, for each element b_i, $o(b_i) = p_i^{m_i}$ and $h_{p_i}^*(b_i) \geq n_i$, one can find a cyclic direct summand C_i such that $|C_i| \geq p_i^{m_i+n_i}$ (Lemma 4.1). Let c_i be one of the generators of the group $C_i\,(i=\overline{1,\,k})$, then $\mathbb{H}(a) < \mathbb{H}(p_i^{n_i}c_i) \leq \mathbb{H}(b_i)$, and, therefore, $\mathbb{H}(a) < \mathbb{H}(p_1^{n_1}c_1 + \ldots + p_k^{n_k}c_k) \leq \mathbb{H}(b)$. The equations $p_i^{n_i}x = a$ $(i=\overline{1,\,k})$ are solvable in the group $< a >_*$. Let $x_i\,(i=\overline{1,\,k})$ be solutions of these equations and, consequently, $h_{p_i}(x_i) = 0$. We can construct homomorphisms $\psi_i :< a >_* \longrightarrow C_i$ $(i=\overline{1,\,k})$ such that $\psi_i(x_i) = c_i$. Since $< a >_*$ is a group of rank 1, there exist integers m and n such that $mx_i = nd$ and $(m,\,n) = 1$ for any $d \in< a >_*$. Taking into account that x_i has zero height with respect to a prime number p_i in the group $< a >_*$, we obtain $(n,\,p_i) = 1$. Now we put $\psi_i(d) = \frac{m}{n}\,c_i$. It is easy to see that the mapping $\psi_i\,(i=\overline{1,\,k})$ is a homomorphism. Besides, the homomorphism $\psi = \psi_1 + \ldots + \psi_k$ maps the element a into the element $p_1^{n_1}c_1 + \ldots + p_k^{n_k}c_k$. Since $C = \bigoplus_{i=1}^{k} C_i$ is an algebraically compact and, consequently, purely injective group, there exists a homomorphism $\alpha \in \mathrm{Hom}(A,\,C)$ such that $\alpha i = \psi$, where $i :< a >_* \longrightarrow A$ is an embedding and $\alpha(a) = p_1^{n_1}c_1 + \ldots + p_k^{n_k}c_k$. Since $\mathbb{H}(p_1^{n_1}c_1 + \ldots + p_k^{n_k}c_k) \leq \mathbb{H}(b)$ and B are fully transitive groups, one can find $\eta \in \mathrm{E}\,(B)$ such that $\eta(p_1^{n_1}c_1 + \ldots + p_k^{n_k}c_k) = b$. Then the homomorphism $\eta\alpha \in \mathrm{Hom}(A,\,B)$ maps the element a into the element b.

Proposition 4.3. *Let the system $\{G_i\}_{i\in I}$ satisfy the condition of finiteness for height matrices. If for any prime number p there exists an element $g_j \in G_j$ $(j \in I)$ such that $o(g_j) = \infty$ and $h_p^*(p^k g_j) \neq \infty$ for any $k \in \mathbb{N}$, then $T_p(\prod_{i\in I} G_i) = \prod_{i\in I} T_p(G_i)$.*

Proof. It is easy to show that

$$T_p(\prod_{i\in I} G_i) = T_p(\prod_{i\in I} T_p(G_i)) \qquad (*)$$

for any prime number p. Let the condition of the proposition be satisfied, i.e., there exists an element $g_j \in G_j$ $(j \in I)$ such that $o(g_j) = \infty$ and $h_p^*(p^k g_j) \neq \infty$ for any $k \in \mathbb{N}$ and a prime number p. The equality $(*)$ implies $T_p(\prod_{i\in I} G_i) \subseteq \prod_{i\in I} T_p(G_i)$. It remains to show that $\prod_{i\in I} T_p(G_i)$ is a torsion group. Assume the contrary, i.e., let there be an element $a \in \prod_{i\in I} T_p(G_i)$ of infinite order. Then one can find coordinates a_α element a such that

$$\mathbb{H}(a) = \inf{}_{\mathfrak{M}}\{\mathbb{H}(a_\alpha)\}_{\alpha\in J\,,J\subseteq I,\,|J|=\aleph_0}\,.$$

As follows from Lemma 4.1, a cyclic direct summand $B_\alpha \subseteq T_p(G_\alpha)$, for which $|B_\alpha| \geq p^{k(\alpha)}$ can be found for any coordinate $a_\alpha\,(\alpha \in J)$ of the element a such that $o(a_\alpha) = p^{k(\alpha)}$. Let b_α be one of the generators of the group $B_\alpha\,(\alpha \in J)$, then $\inf\{H_p(b_\alpha)\}_{\alpha\in J} = (0,\,1,\,2,\,\dots)$. We have $\mathbb{H}(g_j) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(pg_j),\,\mathbb{H}(b_\alpha)\}_{\alpha\in J}$. So, since the system $\{G_i\}_{i\in I}$ satisfies the condition of finiteness, there exists a finite subsystem $\{c_\tau\}_{\tau=\overline{1,\,n}} \subset \{pg_j,\,b_\alpha\}_{\alpha\in J}$ such that $\mathbb{H}(g_j) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(c_\tau)\}_{\tau=\overline{1,\,n}}$. If $pg_j \notin \{c_\tau\}_{\tau=\overline{1,\,n}}$, the previous inequality is impossible. Indeed, the group $\prod\limits_{i\in I} T_p(G_i)$ contains an element b' such that all its non-zero coordinates are elements of $c_\tau\,(\tau = \overline{1,\,n})$. Since the element b' is of finite order and

$$\mathbb{H}(b') = \inf{}_{\mathfrak{M}}\{\mathbb{H}(c_\tau)\}_{\tau=\overline{1,\,n}}\,,\text{then } \mathbb{H}(g_j) \ngeq \inf{}_{\mathfrak{M}}\{\mathbb{H}(c_\tau)\}_{\tau=\overline{1,\,n}}\,.$$

Let $pg_j \in \{c_\tau\}_{\tau=\overline{1,\,n}}$ and $pg_j = c_n$. Since all the elements $c_\tau\,(\tau = \overline{1,\,n-1})$ are of finite order, there exists $m \in \mathbb{N}$ such that $p^m = \max\{o(c_\tau)\}_{\tau=\overline{1,\,n-1}}$. Then $H_p(p^l g_j) < \inf\{H_p(p^l c_\tau)\}_{\tau=\overline{1,\,n}} = H_p(p^{l+1}g_j)$ for any $l \geq m$. Therefore, $\mathbb{H}(g_j) \ngeq \inf_{\mathfrak{M}}\{\mathbb{H}(c_\tau)\}_{\tau=\overline{1,\,n}}$.

Corollary 4.4. *Let the system $\{G_i\}_{i\in I}$ satisfy the condition of finiteness for height matrices. If a torsion free direct summand A_j of the group G_j exists for a certain $j \in I$, then $T_p(\prod\limits_{i\in I} G_i) = \prod\limits_{i\in I} T_p(G_i)$ for any $p \in \pi(A_j)$.*

Lemma 4.5. *Let the system $\{G_\alpha\}_{\alpha\in\mathfrak{A}}$ satisfy the condition of finiteness, $\mathfrak{A} = I \cup J$, and let the following condition be satisfied:*
1) $G_\alpha = A_\alpha \oplus B_\alpha$, if $\alpha \in I \cap J$;
2) $G_\alpha = A_\alpha$, if $\alpha \in \mathfrak{A} \setminus J$;
3) $G_\alpha = B_\alpha$, if $\alpha \in \mathfrak{A} \setminus I$,
where $0 \neq A_\alpha$ is a torsion free group, $0 \neq B_\alpha = T(G_\alpha)$ for any $\alpha \in \mathfrak{A}$. If $\prod\limits_{j\in J} B_j$ is a fully transitive group, the system $\{\prod\limits_{i\in I} A_i,\,\prod\limits_{j\in J} B_j\}$ satisfies the condition of monotonicity.

Proof. Let $A = \prod\limits_{i\in I} A_i$ and $B = \prod\limits_{j\in J} B_j$. Let's consider the first case. Let $a,\,c \in A$, $b \in B$ and $\mathbb{H}(a) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(c),\,\mathbb{H}(b)\}$. Then, as follows from Corollary 4.4, for each prime number p such that $pA \neq A$ there exists $n \in \mathbb{N}$ such that $p^n b = 0$. Therefore, $H_p(p^n a) \geq \inf\{H_p(p^n c),\,H_p(p^n b)\} = H_p(p^n c)$, and so $\mathbb{H}(a) \geq \mathbb{H}(c)$. Let's consider the second case. Let $a \in A$, $b,\,c \in B$ and $\mathbb{H}(b) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a),\,\mathbb{H}(c)\}$, where $\mathbb{H}(b) \ngeq \mathbb{H}(a)$ and $\mathbb{H}(b) \ngeq \mathbb{H}(c)$.

Let 1) $o(b) < \infty$, $o(c) < \infty$ and $b = b_{p_1} + \ldots + b_{p_n}$, where $b_{p_\tau} \in B_{p_\tau}$ and B_{p_τ} ($\tau = \overline{1,\, k}$) are p-components of the torsion subgroup of the group B. Consider all the elements $b_q \in B_q$ such that $q \in \{p_\tau\}_{\tau=\overline{1,k}}$ and $\mathbb{H}(b_q) \ngeq \mathbb{H}(a)$, $\mathbb{H}(b_q) \ngeq \mathbb{H}(c)$. Since $\mathbb{H}(b_q) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a),\, \mathbb{H}(c)\}$ and $\mathbb{H}(b_q) \ngeq \mathbb{H}(a)$, $\mathbb{H}(b_q) \ngeq \mathbb{H}(c)$, we have 1') $H_q(b_q) \gtreqless H_q(a)$ and 2') $H_q(a) \gtreqless H_q(c)$. Let's show that 1') is satisfied. Indeed, since $o(a) = \infty$, $o(b_q) < \infty$ and $\mathbb{H}(b_q) \ngeq \mathbb{H}(a)$, we have $H_q(b_q) \gtreqless H_q(a)$. Let's show that the condition 2') is satisfied. Since $o(a) = \infty$, $o(c) < \infty$, $h_q(a) \neq \infty$ and $h_q(c) \neq \infty$, only two cases are possible: $H_q(a) < H_q(c)$ or $H_q(a) \gtreqless H_q(c)$. Assume $H_q(a) < H_q(c)$, then $\inf\{H_q(a),\, H_q(c)\} = H_q(a) \leq H_q(b_q)$, what contradicts 1'). Let

$$H_q(b_q) = (\delta_0^{(q)},\, \ldots\,,\, \delta_{m(q)}^{(q)},\, \infty,\, \ldots), \qquad (**)$$

$H_q(a) = (n_0^{(q)},\, n_1^{(q)},\, \ldots)$, $H_q(c) = (\sigma_0^{(q)},\, \ldots\,,\, \sigma_{\nu(q)}^{(q)},\, \infty,\, \ldots)$ for all such q. Since $\mathbb{H}(b_q) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a),\, \mathbb{H}(c)\}$ and $H_q(b_q) \gtreqless H_q(a)$; $H_q(a) \gtreqless H_q(c)$, we have $h_q(a) > h_q^*(b_q) \geq h_q^*(c)$ (it means that $n_0^{(q)} > \delta_0^{(q)} \geq \sigma_0^{(q)}$). Since $H_q(b_q) \gtreqless H_q(a)$, there exists the least non-negative integer $r(q)$ such that $n_{r(q)}^{(q)} > \delta_{r(q)}^{(q)} \geq \sigma_{r(q)}^{(q)}$. However, $\delta_{r(q)+1}^{(q)} \geq n_{r(q)+1}^{(q)}$ and $\delta_{r(q)+1}^{(q)} \neq \infty$ (if we assume that $\delta_{r(q)+1}^{(q)} = \infty$, we have $\mathbb{H}(b_q) \geq \mathbb{H}(c)$ what is impossible). We can show that there is a jump between $\delta_{r(q)}^{(q)}$ and $\delta_{r(q)+1}^{(q)}$. Indeed, since $\delta_{r(q)}^{(q)} < n_{r(q)}^{(q)}$, we have $\delta_{r(q)}^{(q)}+1 < n_{r(q)}^{(q)}+1 = n_{r(q)+1}^{(q)} \leq \delta_{r(q)+1}^{(q)}$. If there is a jump between $\delta_\gamma^{(q)}$ and $\delta_{\gamma+1}^{(q)}$, each $\delta_\gamma^{(q)}$th Ulm-Kaplansky invariant of the group B_q is different from xero in the characteristic function. So the sequence $(**)$ is the characteristic function of a certain element $d_q \in B_q$ [47, Lemma 65.3]. On the other hand, the element $q^{m(q)}b_q$ is of order q and its height is not less than $n_{m(q)}^{(q)}$. By Lemma 4.1, the group B_q has a cyclic direct summand F_q whose order is not less than $q^{n_{m(q)}^{(q)}+1}$. Since $n_{m(q)}^{(q)} = n_0^{(q)} + m(q)$, we have $q^{n_{m(q)}^{(q)}+1} = q^{n_0^{(q)}+m(q)+1}$. Let f_q be one of generators of the group F_q. Then the element $q^{n_0^{(q)}} f_q$ has the characteristic function $H_q(q^{n_0^{(q)}} f_q) = (n_0^{(q)},\, \ldots\,,\, n_\lambda^{(q)},\, \infty,\, \ldots)$, where $\lambda \geq m(q)$.

Thus, we obtain that $H_q(d_q) > H_q(c)$ and $H_q(q^{n_0^{(q)}} f_q) > H_q(a)$. It follows that $\mathbb{H}(d_q) > \mathbb{H}(c)$ and $\mathbb{H}(q^{n_0^{(q)}} f_q) > \mathbb{H}(a)$. On the other hand, since $h_q(q^\alpha d_q) < h_q(q^{\alpha+n_0^{(q)}} f_q)$ for any $0 \leq \alpha \leq r(q)$ and $h_q(q^\alpha d_q) > h_q(q^{\alpha+n_0^{(q)}} f_q)$ for all $\alpha > r(q)$, we have $H_q(d_q + q^{n_0^{(q)}} f_q) = \inf\{H_q(d_q),\, H_q(q^{n_0^{(q)}} f_q)\}$. Here, as it can be seen from the construction of the element $d_q + q^{n_0^{(q)}} f_q$, $\mathbb{H}(d_q +$

$q^{n_0^{(q)}} f_q) \leq \mathbb{H}(b_q)$. Since B_q is a fully transitive group, there exists $\varphi_q \in \mathrm{E}\,(B_q)$ such that $\varphi_q(d_q) + \varphi_q(q^{n_0^{(q)}} f_q) = b_q$. It follows that $\mathbb{H}(\varphi_q(d_q)) > \mathbb{H}(c)$, $\mathbb{H}(\varphi_q(q^{n_0^{(q)}} f_q)) > \mathbb{H}(a)$. Further, let's take the equality $b = b_{p_1} + \ldots + b_{p_k}$ and replace all the elements b_q with the property $\mathbb{H}(b_q) \ngeq \mathbb{H}(a)$ and $\mathbb{H}(b_q) \ngeq \mathbb{H}(c)$ by their above-mentioned decompositions. We obtain that $b = b'_1 + \ldots + b'_n$, where $n \geq k$. Thus, for any $\beta = \overline{1,\, n}$ there is an element $v \in \{a,\, c\}$ such that $\mathbb{H}(b'_\beta) \geq \mathbb{H}(v)$.

2) Let $o(b) < \infty$, $o(c) = \infty$ and $\{c_j\}_{j\in J}$, $\{a_i\}_{i\in I}$ be coordinates of the elements c and a respectively. Since $\mathbb{H}(b) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a),\, \mathbb{H}(c)\}$, we obtain $\mathbb{H}(b) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(a_i),\, \mathbb{H}(c_j)\}_{i\in I,\, j\in J}$. The relation $o(b) < \infty$, implies existence of a finite number of elements $\{c_{j_k},\, a_{i_l}\}_{k=\overline{1,\,r},\, l=\overline{1,\,t}} \subset \{c_j\}_{j\in J} \cup \{a_i\}_{i\in I}$ such that

$$\mathbb{H}(b) \geq \inf{}_{\mathfrak{M}}\{\mathbb{H}(a_{i_l}),\, \mathbb{H}(c_{j_k})\}_{k=\overline{1,\,r},\, l=\overline{1,\,t}}\,.$$

The set $\{c_{j_k},\, a_{i_l}\}_{k=\overline{1,\,r},\, l=\overline{1,\,t}}$ cannot consist only of the elements c_{j_k} $(k = \overline{1,\, r})$ or only of elements a_{i_l} $(l = \overline{1,\, t})$. Indeed, if $a_{i_l} = 0$ for any $l = \overline{1,\, t}$, we obtain $\mathbb{H}(b) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(c_{j_k})\}_{k=\overline{1,\,r}} \geq \inf_{\mathfrak{M}}\{\mathbb{H}(c_j)\}_{j\in J} = \mathbb{H}(c)$ what is impossible because $\mathbb{H}(b) \ngeq \mathbb{H}(c)$ by condition. Similarly, one can verify the condition $c_{j_k} = 0$ for any $k = \overline{1,\, r}$. Let $\rho_{j_k} : B_{j_k} \longrightarrow B\,(k = \overline{1,\, r})$ and $\rho_{i_l} : A_{i_l} \longrightarrow A\,(l = \overline{1,\, t})$ be embeddings. Then $\mathbb{H}(b) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(\rho_{j_1}(c_{j_1}) + \ldots + \rho_{j_k}(c_{j_k})),\, \mathbb{H}(\rho_{i_1}(a_{i_1}) + \ldots + \rho_{i_t}(a_{i_t}))\}$. The element $\rho_{j_1}(c_{j_1}) + \ldots + \rho_{j_k}(c_{j_k})$ is of finite order. So, as follows from the case 1), there exists a finite number of elements $b_1, \ldots, b_m \in B$ such that $b = b_1 + \ldots + b_m$, and for each element b_β $(\beta = \overline{1,\, m})$, at least one of the inequalities must be valid: $\mathbb{H}(b_\beta) \geq \mathbb{H}(\rho_{j_1}(c_{j_1}) + \ldots + \rho_{j_k}(c_{j_k}))$ or $\mathbb{H}(b_j) > \mathbb{H}(\rho_{i_1}(a_{i_1}) + \ldots + \rho_{i_t}(a_{i_t}))$. Then for each $\beta = \overline{1,\, m}$ we obtain at least one of the inequalities $\mathbb{H}(b_\beta) \geq \mathbb{H}(c)$ or $\mathbb{H}(b_\beta) > \mathbb{H}(a)$.

3) Let $o(b) = \infty$, $o(c) = \infty$. Among the coordinates $\{b_j\}_{j\in J}$ of the element b we choose only those for which $\mathbb{H}(b_j) \ngeq \mathbb{H}(c)$ (they exist because otherwise $\mathbb{H}(b) \geq \mathbb{H}(c)$, what contradicts the supposition $\mathbb{H}(b) \ngeq \mathbb{H}(c)$). As follows from 1) and 2), since $o(b_j) < \infty$ for any $j \in J$, there exist elements $\{b_k^{(j)}\}_{k=\overline{1,\,r}}$, $b_k^{(j)} \in B$ such that $\rho_j(b_j) = b_1^{(j)} + \ldots + b_r^{(j)}$. Here $\rho_j : B_j \longrightarrow B$ is an embedding and for any $k = \overline{1,\, r}$ at least one of the following inequalities must be valid: $\mathbb{H}(b_k^{(j)}) > \mathbb{H}(c)$ or $\mathbb{H}(b_k^{(j)}) > \mathbb{H}(a)$. Let $\pi_j : B \longrightarrow B_j$ be a projection, then $b_j = \pi_j(b_1^{(j)}) + \ldots + \pi_j(b_r^{(j)})$ and $\mathbb{H}(\pi_j(b_k^{(j)})) > \mathbb{H}(c)$ or $\mathbb{H}(\pi_j(b_k^{(j)})) > \mathbb{H}(a)$ for any $k = \overline{1,\, r}$. Then each element b_j can be represented in the form $b_j = b_{j_1} + b_{j_2}$, where b_{j_1} equals the sum of elements $\pi_j(b_k^{(j)})$

such that $\mathbb{H}(\pi_j(b_k^{(j)})) > \mathbb{H}(c)$, and b_{j_2} equals the sum of elements $\pi_j(b_k^{(j)})$, for which $\mathbb{H}(\pi_j(b_k^{(j)})) \not> \mathbb{H}(c)$. We obtain $\mathbb{H}(b_{j_1}) > \mathbb{H}(c)$ and $\mathbb{H}(b_{j_2}) > \mathbb{H}(a)$. Then $b = (\dots, b_{j_1} + b_{j_2}, \dots)$, and, for some $b_j \neq 0$, $b_{j_1} = 0$ or $b_{j_2} = 0$ can take place. Therefore, the element b can be written in the form $b = b_1 + b_2$, where $b_1 = (\dots, b_{j_1}, \dots)$ and $b_2 = (\dots, b_{j_2}, \dots)$. It follows that $\mathbb{H}(b_1) \geq \mathbb{H}(c)$ and $\mathbb{H}(b_2) > \mathbb{H}(a)$.

Theorem 4.6. *Let the system $\{G_\alpha\}_{\alpha \in \mathfrak{A}}$ satisfy the condition of finiteness, $\mathfrak{A} = I \cup J$, and the following conditions are satisfied:*
1) $G_\alpha = A_\alpha \oplus B_\alpha$, if $\alpha \in I \cap J$;
2) $G_\alpha = A_\alpha$, if $\alpha \in \mathfrak{A} \setminus J$;
3) $G_\alpha = B_\alpha$, if $\alpha \in \mathfrak{A} \setminus I$,
where $0 \neq A_\alpha$ is a torsion free group, $0 \neq B_\alpha = T(G_\alpha)$ for any $\alpha \in \mathfrak{A}$. The group $G = \prod\limits_{\alpha \in \mathfrak{A}} G_\alpha$ is fully transitive if and only if $\prod\limits_{i \in I} A_i$ and $\prod\limits_{j \in J} B_j$ are fully transitive.

Proof. Let $A = \prod\limits_{i \in I} A_i$ and $B = \prod\limits_{j \in J} B_j$. Then, as follows from Theorem 1.3, these groups are fully transitive. Conversely. Let's show that the system of the groups $\{A,\, B\}$ is fully transitive. The system $\{G_\alpha\}_{\alpha \in \mathfrak{A}}$ satisfies the condition of finiteness. So, as follows from Corollary 4.4, $T_p(B) = \prod\limits_{j \in J} B_{j_p}$ for any prime number $p \in \pi(A)$, where B_{j_p} $(j \in J)$ is the p-component of the group B_j. Then $H_p(b) \not\leq H_p(a)$ for each prime number $p \in \pi(A)$. It follows that $\mathbb{H}(b) \not\leq \mathbb{H}(a)$, i.e., only the following inequality is possible: $\mathbb{H}(b) > \mathbb{H}(a)$. By Lemma 4.2, there exists $\varphi \in \mathrm{Hom}(A,\, B)$ such that $\varphi(a) = b$. The condition of monotonicity for the system $\{A,\, B\}$ follows from Lemma 4.5.

The following result is obtained directly from this Theorem for the class of split groups.

Corollary 4.7. *Let the system $\{G_\alpha\}_{\alpha \in \mathfrak{A}}$ of split groups satisfy the conditio of finiteness. The group $G = \prod\limits_{\alpha \in \mathfrak{A}} G_\alpha$ is fully transitive if and only if the groups $\prod\limits_{\alpha \in \mathfrak{A}} T(G_\alpha)$ and $\prod\limits_{\alpha \in \mathfrak{A}} (G_\alpha \,/\, T(G_\alpha))$ are fully transitive.*

Corollary 4.8. *A split group is fully transitive if and only if its torsion subgroup and its torsion free subgroup are fully transitive.*

Some classes of fully transitive split groups are obtained in the following corollary.

Corollary 4.9. *Let $G = A \oplus B$ where $B = T(G)$ and let the groups A and B satisfy at least one of the following conditions:*
1) A is a separable group such that $\pi(C) \cap \pi(K) = \emptyset$ takes place for all non-isomorphic rank 1 direct summands C and K of the group A;
2) A is an algebraically compact group;
3) B is a separable group;
4) B is a totally projective group;
5) B is a direct sum of a separable and a totally projective group. Then G is a fully transitive group.

Proof. Fully transitivity of groups from 1), 2), and 5) follows from Theorem 2.4, Corollary 3.9, and Corollary 3.8; fully transitivity of groups from 3) and 4) is obtained from [47].

We recall that a subgroup H of a group G is said to be *absorbing* in G [25], if $T(G/H) = 0$.

Let $\mathfrak{B}$ denote the class of all reduced groups in which each absorbing subgroup is a direct summand.

Corollary 4.10. *A group $G \in \mathfrak{B}$ is fully transitive if and only if $T(G)$ is fully transitive.*

Proof. As follows from [25] and [26], $G = T(G) \oplus F$, where F is a direct sum of a finite number of mutually isomorphic torsion free rank 1 groups; it remains to apply Corollary 4.8 and Theorem 2.4.

The following corollaries demonstrate some examples of classes of fully transitive groups.

Let $\mathfrak{C}$ denote the class of all reduced groups in which each isotype (pure) subgroup is a direct summand.

Corollary 4.11. *The class $\mathfrak{C}$ consists of fully transitive groups.*

Proof. As follows from [2, Theorem 2], an arbitrary group $G \in \mathfrak{C}$ has the form $G = T(G) \oplus F$, where the p-component $T(G)$ is bounded and the group F is a direct sum of a finite number of mutually isomorphic rank 1 groups. Applying Corollary 4.9, we obtain fully transitivity of the group G.

We recall that a subgroup B is said to be *balanced* in a group A [27] if each residue class $a + B$ contains an element x such that $\mathbb{H}_A(x) = \mathbb{H}_{A/B}(a + B)$ and $o(x) = o(a + B)$.

Let $\mathfrak{S}$ denote the class of all reduced groups in which each pure subgroup is balanced.

Corollary 4.12. *The class* $\mathfrak{S}$ *consists of fully transitive groups.*

Proof. As follows from [27, Proposition 4.8], each group from the class $\mathfrak{S}$ has the form $G = A \oplus B$, where B is a torsion group in which each p-component is bounded and A is a homogeneous completely decomposable torsion free finite rank group. Therefore, fully transitivity of the group G follows from Corollary 4.9.

Theorem 4.13. *Let* $\{G_\alpha\}_{\alpha \in \mathfrak{A}}$ *be a system of* $\mathbf{s}$*-generally slender groups and* $\mathfrak{A} = I \cup J$*, and let the following conditions be satisfied:*
1) $G_\alpha = A_\alpha \oplus B_\alpha$ *if* $\alpha \in I \cap J$*;*
2) $G_\alpha = A_\alpha$ *if* $\alpha \in \mathfrak{A} \setminus J$*;*
3) $G_\alpha = B_\alpha$ *if* $\alpha \in \mathfrak{A} \setminus I$*,*
where $0 \neq A_\alpha$ *is a torsion free group,* $0 \neq B_\alpha = T(G_\alpha)$ *for any* $\alpha \in \mathfrak{A}$*. The group* $\prod\limits_{\alpha \in \mathfrak{A}} G_\alpha$ *is fully transitive if and only if the following conditions are satisfied:*
1) $\prod\limits_{\alpha \in I} G_\alpha$ *is a fully transitive group;*
2) $\{G_\alpha\}_{\alpha \in J}$ *is a fully transitive system;*
3) if $p \in \pi(\prod\limits_{\alpha \in I} G_\alpha)$, $T_p(\prod\limits_{\alpha \in J} G_\alpha) = \prod\limits_{\alpha \in J} T_p(G_\alpha)$.

Proof. By assumption, we obtain that $G = \prod\limits_{\alpha \in \mathfrak{A}} G_\alpha = \prod\limits_{\alpha \in I} G_\alpha \oplus \prod\limits_{\alpha \in J} G_\alpha$. Fully transitivity of the group G implies fully transitivity of the groups $\prod\limits_{\alpha \in I} G_\alpha$

and $\prod\limits_{\alpha\in J} G_\alpha$ (Theorem 1.3). By Proposition 1.4, the system $\{G_\alpha\}_{\alpha\in J}$ is fully transitive. Since $\{G_\alpha\}_{\alpha\in\mathfrak{A}}$ is a system of **s**-generally slender groups and G is a fully transitive group, the system $\{G_\alpha\}_{\alpha\in\mathfrak{A}}$ satisfies the condition of finiteness by Theorem 3.1. By Corollary 4.4, the condition 3) is satisfied. Conversely. By virtue of Corollary 3.7, the group $\prod\limits_{\alpha\in J} G_\alpha$ is fully transitive. Let's show that the system $\{G_\alpha\}_{\alpha\in\mathfrak{A}}$ satisfies the condition of finiteness. Consider an arbitrary torsion free group $A_i \in \{G_\alpha\}_{\alpha\in\mathfrak{A}}$ and an arbitrary element $a_i \in A_i$ such that $\mathbb{H}(a_i) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(g_\alpha)\}_{\alpha\in\mathfrak{A}'}$, where $\mathfrak{A}' \subseteq \mathfrak{A}$ and $|\mathfrak{A}'| = \aleph_0$. Let's take all prime numbers p for which $h_p(a_i) \neq \infty$ in the group A_i. Then $T_p(\prod\limits_{\alpha\in J} G_\alpha) = \prod\limits_{\alpha\in J} T_p(G_\alpha)$ and for any such prime p there exists $m(p) \in \mathbb{N}$ such that $h_p(p^{m(p)}a_i) \geq h_p(p^{m(p)}g_\alpha^{(p)})$ for a certain element $g_\alpha^{(p)} \in \{g_\alpha\}_{\alpha\in\mathfrak{A}'}$ of infinite order. Then $H_p(a_i) \geq H_p(g_\alpha^{(p)})$ for all such p and all such elements $g_\alpha^{(p)}$. It follows that $\mathbb{H}(a_i) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(g_\alpha^{(p)})\}_{\alpha\in\mathfrak{A}'\cap J}$. Since the system $\{G_\alpha\}_{\alpha\in I}$ consists of **s**-generally slender and since $\prod\limits_{\alpha\in I} G_\alpha$ is a fully transitive group, the system $\{G_\alpha\}_{\alpha\in I}$ satisfies the condition of finiteness by Theorem 3.1. Then its subsystem $\{G_\alpha\}_{\alpha\in\mathfrak{A}'\cap I}$ also satisfies the condition of finiteness. Therefore, among the elements $\{g_\alpha^{(p)}\}_{\alpha\in\mathfrak{A}'\cap J}$ one can find a finite subsystem of elements $\{g_{\alpha r}^{(p)}\}_{r=\overline{1,n}}$, such that $\mathbb{H}(a_i) \geq \inf_{\mathfrak{M}}\{\mathbb{H}(g_{\alpha r}^{(p)})\}_{r=\overline{1,n}}$. Thus we see that the system $\{G_\alpha\}_{\alpha\in\mathfrak{A}}$ satisfies the conditio of finiteness. It means that the group G is fully transitive by Theorem 4.6.

If **s**-generally slender groups G_i $(i \in I)$ are mixed separable groups [22], the conditions 1) – 3) of Theorem 3.1 can be replaced by more easy-to-interpret ones.

Theorem 4.14. *Let $\{G_i\}_{i\in I}$ be a family of separable groups. The group $\prod\limits_{i\in I} G_i$ is fully transitive if and only if the following conditions are satisfied:*
1) if for a prime number p there exists a torsion free direct summand C from G such that $pC \neq C$, then $T_p(G) = \prod\limits_{i\in I} T_p(G_i)$;
2) $\pi(A) \cap \pi(B) = \emptyset$ takes place for all non-isomorphic torsion free rank 1 direct summands A and B from G.

Proof. Since separable groups are **s**-generally slender, the system $\{G_i\}_{i\in I}$

satisfies the condition of finiteness by Theorem 3.1. Then the condition 1) is satisfied by Theorem 4.3. Validity of the condition 2) follows from Lemmas 2.2 and 2.1. Conversely. Let a, $b \in G$ and $\mathbb{H}(a) \leq \mathbb{H}(b)$, where $a = (\dots, a_i, \dots)$, $b = (\dots, b_i, \dots)$. The elements a_i, $b_i \in G_i$ can be embedded into a completely decomposable direct summand G'_i of the group G_i for each $i \in I$, i.e., $G'_i = A_i \oplus B_i$, where A_i is a completely decomposable torsion free finite rank group, and B_i is a direct sum of a finite number of cyclic p-groups. It means that the elements a and b belong to the direct summand $G' = \prod\limits_{i\in I} A_i \oplus \prod\limits_{i\in I} B_i$ of the group G. So, to prove existence of $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = b$, it is sufficient to show that G' is a fully transitive group. By virtue of the previous Theorem, fully transitivity of the group G' is seen from the fact that $\prod\limits_{i\in I} A_i$ is a fully transitive group and $\{B_i\}_{i\in I}$ is a fully transitive system. Fully transitivity of the latter follows from Corollaries 3.9 and 3.7. Fully transitivity of $\prod\limits_{i\in I} A_i$ is shown in Corollary 2.7.

Corollary 4.15. *A separable abelian group is fully transitive if and only if $\pi(A) \cap \pi(B) = \emptyset$ takes place for all its non-isomorphic torsion free rank 1 direct summands A and B.*

Below we study split direct product of **s**-generally slender groups in connection with the property of fully transitivity.

Since a mixed group is split if and only if its reduced subgroup is split, the groups in the following proposition and its corollaries are supposed to be reduced.

Proposition 4.16. *Let $\prod\limits_{i\in I} G_i$ be a mixed group. The group G is split if and only if the following conditions are satisfied:*
1) if G_i $(i \in I)$ is a mixed group, it is split;
2) $T(\prod\limits_{i\in I} G_i) = \prod\limits_{i\in I} T(G_i)$.

Proof. The condition 1) follows from [47, p. 224]. Let's prove 2). Suppose the contrary, i.e., let $T(\prod\limits_{i\in I} G_i) \neq$
$\prod\limits_{i\in I} T(G_i)$. Then the group $\prod\limits_{i\in I} T(G_i)$ contains an element g such that $o(g) = \infty$, $g = (\dots, g_i, \dots)$ and $o(g_i) = n_i$ $(n_i \in \mathbb{N})$ for any $i \in I$. Consider an arbitrary coordinate $g_i \neq 0$ and let $n_i = p_i^{k_i} m_i$, where p_i is a prime number such that

$(p_i, m_i) = 1$ and $k_i, m_i \in \mathbb{N}$. Then the element $m_i g_i$ is of order $p_i^{k_i}$. The group G_i, as seen from Lemma 4.1, contains a cyclic direct summand A_i of order not less than $p_i^{k_i}$. Since $o(g) = \infty$, one can find a coordinate g_j of the element g and a prime number p_j such that $o(g_j) = p_j^{k_j} m_j$, where $(p_j, m_j) = 1$ and $k_j, m_j \in \mathbb{N}$, $p_j^{k_j} > p_i^{k_i}$. Then, according to Lemma 4.1, the group G_j contains a cyclic direct summand A_j of order not less than $p_j^{k_j}$. Continuing in such a way, we obtain a system of groups $\{A_i\}_{i \in I'}$, where $I' \subseteq I$ and $|I'| = \aleph_0$, whose orders are collectively unbounded. Since each group $A_i \, (i \in I')$ is a direct summand in the group G_i, we have $\prod\limits_{i \in I} G_i = \prod\limits_{i \in I \setminus I'} G_i \oplus \prod\limits_{i \in I'} B_i \oplus \prod\limits_{i \in I'} A_i$, where $G_i = A_i \oplus B_i$ for any $i \in I'$. Since $\prod\limits_{i \in I} G_i$ is split, each direct summand of this group is also spilt, i.e., $\prod\limits_{i \in I'} A_i = A \oplus T(\prod\limits_{i \in I'} A_i)$, where A is a torsion free group. However, $T(\prod\limits_{i \in I'} A_i)$ is a bounded group [46, Corollary 40.3], what contradicts the fact that the group contains an unbounded subgroup $\bigoplus\limits_{i \in I'} A_i$. Conversely. Since the condition 1) is satisfied, we have $G_i = A_i \oplus T(G_i) \, (i \in I)$, where A_i is a torsion free group. Then $\prod\limits_{i \in I} G_i = \prod\limits_{i \in I} A_i \oplus \prod\limits_{i \in I} T(G_i) = \prod\limits_{i \in I} A_i \oplus T(\prod\limits_{i \in I} G_i)$.

Corollary 4.17. *If $G = \prod\limits_{i \in I} G_i$ isa mixed group, where $G_i \, (i \in I)$ are torsion groups, the group G is not split.*

Corollary 4.18. *Let $G = \prod\limits_{i \in I} G_i$ be a mixed group, $\{G_i\}_{i \in I}$ be a system of groups satisfying the condition of finiteness. The group G is split if and only if the following conditions are satisfied:*
1) if $G_i \, (i \in I)$ is a mixed group, it is split;
2) if for a prime number p one can find such a torsion free direct summand A of the group G that $pA = A$, then $T_p(\prod\limits_{i \in I} G_i) = \prod\limits_{i \in I} T_p(G_i)$.
3) $\bigoplus\limits_{p \in \pi} \prod\limits_{i \in I} T_p(G_i) = \prod\limits_{i \in I} \bigoplus\limits_{p \in \pi} T_p(G_i)$.

Proof. Necessity follows from Proposition 4.16. To prove necessity, we have to demonstrate that $T(\prod\limits_{i \in I} G_i) =$ $\prod\limits_{i \in I} T(G_i)$. Let p be a prime number for which one can find $i \in I$ such that the group $G_i / T(G_i)$ is not p-divisible. Since the system $\{G_i\}_{i \in I}$ satisfies the condition of finiteness, it follows that $T_p(\prod_{i\in I} G_i) = \prod_{i\in I} T_p(G_i)$ by Corollary 4.4. Then we have $T(\prod_{i\in I} G_i) = \bigoplus_{p\in\pi} T_p(\prod_{i\in I} G_i) = \bigoplus_{p\in\pi}\prod_{i\in I} T_p(G_i) = \prod_{i\in I}\bigoplus_{p\in\pi} T_p(G_i) = \prod_{i\in I} T(G_i)$.

Corollary 4.19. *A fully transitive mixed group G equal to a direct product of* **s***-generally slender groups G_i $(i \in I)$ is split if and only if the following conditions are satisfied:*
1) if G_i $(i \in I)$ is a mixed group, it is split;
2) if for a prime number p one can find such a torsion free direct summand A of the group G that $pA = A$, $T_p(\prod_{i\in I} G_i) = \prod_{i\in I} T_p(G_i)$.
3) $\bigoplus_{p\in\pi}\prod_{i\in I} T_p(G_i) = \prod_{i\in I}\bigoplus_{p\in\pi} T_p(G_i)$.

Proof follows from Theorem 3.1 and the previous Corollary.

The following theorem represents influence of fully transitivity on separability of a direct product of abelian reduced groups.

Theorem 4.20. *A group $G = \prod_{i\in I} G_i$ is a fully transitive separable group if and only if the following conditions are satisfied:*
1) G_i is a separable group for any $i \in I$;
2) $\pi(A) \cap \pi(B) = \emptyset$ takes place for all homogeneous torsion free direct summands A and B from G such that $t(A) \neq t(B)$;
3) $G = \bigoplus_{i\in J_1} G_i \oplus \bigoplus_{i\in J_2} A_i \oplus \bigoplus_{j=1}^{k}\prod_{i\in I'} A_i^{(j)} \oplus \prod_{i\in I\setminus J_1} T(G_i)$, where J_1, J_2 are finite sets such that $J_1 \subset I$, $J_2 \subset I\setminus J_1$, $I' = I\setminus(J_1 \cup J_2)$; $\prod_{i\in I\setminus J_1} T(G_i)$ is a bounded group; $A_i \cong G_i/T(G_i)$ for any $i \in I \setminus J_1$; $\prod_{i\in I'} A_i^{(j)}$ $(j = \overline{1, k})$ are homogeneous torsion free groups and if $\prod_{i\in I'} A_i^{(j)} \neq \bigoplus_{i\in I'} A_i^{(j)}$ for a certain $j = \overline{1, k}$, then $\prod_{i\in I'} A_i^{(j)}$ is of idempotent type; $\bigoplus_{i\in J_2} A_i$ is a homogeneously decomposable torsion free group.

Proof. Necessity. The condition 2) follows from Lemma 2.2 and Lemma 2.1. The condition 1) is satisfied due to [42, Corollary 3]. Let's show validity of condition 3). Since G is a separable group, the conditions 2) and 3) from [42, Corollary 3] are satisfied. It means that there exists a finite subset

$J_1 \subset I$ such that the groups $T(G_i)\,(i \in I \setminus J_1)$ are collectively bounded. Then $G = \bigoplus_{i\in J_1} G_i \oplus \prod_{i\in I\setminus J_1} G_i = \bigoplus_{i\in J_1} G_i \oplus \prod_{i\in I\setminus J_1} A_i \oplus \prod_{i\in I\setminus J_1} T(G_i)$, where $A_i \cong G_i/T(G_i)$ for any $i \in I \setminus J_1$ and $\prod_{i\in I\setminus J_1} T(G_i)$ is a bounded torsion group. The finite subset $J_2 \subset I \setminus J_1$ and $k \in \mathbb{N}$ such that $\prod_{i\in I\setminus J_1} A_i = \prod_{i\in J_2} A_i \oplus \prod_{i\in I'} A_i$, where $k = \max|\tau(A_i)|_{i\in I'}$, and $\tau(A_i)\,(i \in I',\ I' = I \setminus (I_1 \cup I_2))$ is the type set of torsion free rank 1 direct summands of the groups A_i, exist due to condition 2) of the Corollary and condition b) from [42, Corollary 3]. Since each group $A_i\,(i \in I')$ is a fully transitive separable torsion free group, it is homogeneously decomposable by Theorem 2.4. Then $\prod_{i\in I'} A_i = \prod_{i\in I'} \bigoplus_{j=1}^{k} A_i^{(j)} = \bigoplus_{j=1}^{k} \prod_{i\in I'} A_i^{(j)}$, and $\prod_{i\in I'} A_i^{(j)}$ is of idempotent type for any $j = \overline{1,\,k}$ such that $\prod_{i\in I'} A_i^{(j)} \neq \bigoplus_{i\in I'} A_i^{(j)}$ [3]. The fact that $\bigoplus_{i\in J_2} A_i$ is a homogeneously decomposable group follows also from its fully transitivity and separability. Sufficiency. Since the conditions of Theorem 4.14 are satisfied, G is a fully transitive group. The group G_i is separable for any $i \in I$. So, to prove separability of the group G, it is sufficient to see that $\prod_{i\in I'} A_i^{(j)}$ is a separable group for any $j = \overline{1,\,k}$ [42, Corollary 3].

Note. If the groups $G_i\,(i \in I)$ in Theorem 4.20 are countable, one can obtain additional information about the groups $\bigoplus_{i\in J_1} G_i \oplus \bigoplus_{i\in J_2} A_i$ and $A_i^{(j)}\,(i \in I',\ j = \overline{1,\,k})$ from the condition 3): these groups are completely decomposable [22, Corollary 1.6]. It follows that if G is a mixed group, it is split.

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