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Quotient divisible and almost completely decomposable groups

Alexander A. Fomin

Abstract. The quotient divisible abelian groups, which are dual to the almost completely decomposable torsion-free abelian groups, are investigated. In particular, the well known example of anomalous direct decompositions by A.L.S. Corner is considered on dual quotient divisible groups.

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1 Introduction

The notion of quotient divisible group has been introduced in [1] as a generalization of two classes of group. The first one is the class $\mathcal{G}$ of honestly mixed groups introduced earlier by S. Glaz and W. Wickless [2], and the second class is the well known class of torsion free quotient divisible groups by R. Beaumont and R. Pierce [3]. The mixed quotient divisible groups are considered also in [4-11]. The main result motivating the introduction of the quotient divisible mixed groups is the duality between the quotient divisible groups and the torsion free groups of finite rank introduced in [1] as well.

The almost completely decomposable groups have been researched by many authors for a long time. We mention contributions of D. Arnold, K. Benabdallah, E.A. Blagoveshenskaya, R. Burkhardt, A.L.S. Corner, M. Dugas, T. Faticoni, L. Fuchs, R. Goebel, B. Jonsson, S.F. Kozhukhov, A. Mader, O. Mutzbauer, E. Lee Lady, F. Loonstra, J. Reid, P. Schultz, C. Vinsonhaler, A.V. Yakovlev, the list is obviously far from being complete.

The main goal of the present paper is an application of the mentioned above duality for investigation of the almost completely decomposable groups. Since the original duality is a duality of categories with quasi-homomorphisms, a direct application is impossible. There is no difference between the almost completely decomposable groups and the completely decomposable groups in such a category. Thus we use a new approach developed in [10].

Every pair consisting of a torsion free group A and a basis (a maximal linearly independent set of elements) $x_1, \dots, x_n$ of A gives a dual pair consisting of a quotient divisible group A^* and a basis $x_1^*, \dots, x_n^*$ of A^* and conversely. It is proved (Corollary 10) that for every almost completely decomposable group A it is possible to choose a basis $x_1, \dots, x_n$ of A such that the quotient divisible group A^* is decomposed into a direct sum of rank-1 quotient divisible subgroups. This is a simplification. Considering a finite extension A of a completely decomposable group B and a common basis $x_1, \dots, x_n$ for two groups, we obtain two dual bases $x_{1B}^*, \dots, x_{nB}^*$ and $x_{1A}^*, \dots, x_{nA}^*$ according to B and to A. They differ by torsion elements $t_1, \dots, t_n$ such that $x^*_{1A} = x^*_{1B} + t_1, \ldots, x^*_{nA} = x^*_{nB} + t_n$. The basis $x^*_{1A}, \ldots, x^*_{nA}$ and therefore the sequence $t_1, \ldots, t_n$ determines completely the group A in this configuration with the fixed basis $x_1, \ldots, x_n$. In such a way we obtain a description of the almost completely decomposable groups (Theorem 15) in terms of the sequences $(t_1, \ldots, t_n)$ of torsion elements. Note that all quotient divisible groups considered in Theorem 15 are decomposed into a direct sum of rank-1 quotient divisible subgroups, while their dual almost completely decomposable groups can be indecomposable (Theorem 11 and Corollary 16) or they can have anomalous direct decompositions as in an example below.

In the final section we apply this description for a dualization of the famous masterpiece by A.L.S. Corner [12]. For every pair $0 < k \leq n$ of integers, there exists a torsion free group C of rank n such that for every decomposition of the number $n = n_1 + \ldots + n_k$ into a sum of k positive integers, the group C can be decomposed into a direct sum of k indecomposable subgroups of ranks $n_1, \ldots, n_k$, respectively.

2 Preliminaries

All groups will be additive abelian groups. Let n be a positive integer and p a prime number, $Z, Q, Z_n = Z/nZ, \widehat{Z}_p$ denote the ring of integers, the field of rational numbers, the ring of residue classes modulo n, the ring of p-adic integers, respectively. $Q_p = \left\{ \frac{m}{n} \in Q \,|g.c.d.\,(n,p) = 1 \right\}$ and $Q^{(p)} = \left\{ \frac{m}{p^n} \in Q \,|m, n \in Z \right\}$. The ring $\widehat{Z} = \prod\limits_p \widehat{Z}_p$ is the Z-adic completion of Z, it is called the ring of *universal integers*. The additive groups of the rings have the same notations.

If $x_1, \ldots, x_n$ are elements of an abelian group A, then $\langle x_1, \ldots, x_n \rangle$ is the subgroup of A generated by these elements, $\langle x_1, \ldots, x_n \rangle_*$ is the pure hull of these elements, that is $a \in \langle x_1, \ldots, x_n \rangle_* \Leftrightarrow$ there exists a nonzero integer m such that $ma \in \langle x_1, \ldots, x_n \rangle$. In particular, all torsion elements of A belong to $\langle x_1, \ldots, x_n \rangle_*$. At last, $\langle x_1, \ldots, x_n \rangle_R$ denotes the submodule of an R-module generated by these elements.

A set of elements $x_1, \ldots, x_n$ of an abelian group (of an R-module) is called *linearly independent over* Z, if every equality $m_1 x_1 + \ldots + m_n x_n = 0$ with integer coefficients implies $m_1 = \ldots = m_n = 0$. A set of elements $x_1, \ldots, x_n$ of a $\widehat{Z}$-module is called *linearly independent over* $\widehat{Z}$, if every equality $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ with universal integer coefficients implies $\alpha_1 x_1 = \ldots = \alpha_n x_n = 0$. In particular, the set $0, \ldots, 0$ is linearly independent over $\widehat{Z}$.

We use the characteristics (m_p) and the types $\tau = [(m_p)]$ in the same manner as in [13] denoting the zero characteristic and the zero type by 0. As usual $(m_p) \geq (k_p)$ if $m_p \geq k_p$ for all prime numbers p. In this case we define $(m_p) - (k_p) = (m_p - k_p)$ setting $\infty - \infty = 0$.

If $\alpha = (\alpha_p) \in \widehat{Z}$, we define the *characteristic of* α as $char\,(\alpha) = (m_p)$, where α_p is divisible by p^{m_p} in $\widehat{Z}_p$ and m_p is the maximal power. If $\alpha_p = 0$ then $m_p = \infty$. Every finitely generated ideal I of the ring $\widehat{Z}$ is of the form $I = I_\chi = \left\{ \alpha \in \widehat{Z} \,|char\,(\alpha) \geq \chi \right\}$ for a characteristic χ. Let $Z_\chi = \widehat{Z}/I_\chi$. As a $\widehat{Z}$-module, Z_χ is cyclic and finitely presented.

A $\widehat{Z}$-module M is called *finitely presented*, if there exists an exact sequence of $\widehat{Z}$-module homomorphisms $\widehat{Z}^m \to \widehat{Z}^n \to M \to 0$ for positive integers m and n. Every finitely presented $\widehat{Z}$-module M is of the form $M \cong Z_{\chi_1} \oplus \ldots \oplus Z_{\chi_n}$. The decomposition is not unique in general, even the number of summands is not an invariant. But it is uniquely definite at the additional condition on the characteristics $\chi_1 \leq \ldots \leq \chi_n$. Every finitely generated submodule N of a finitely presented $\widehat{Z}$-module M is finitely presented and the quotient M/N is finitely presented as well, see [14].

For an element x of a finitely presented $\widehat{Z}$-module M and a prime p, we define: m_p is the greatest nonnegative integer such that p^{m_p} divides x in M and k_p is the least nonnegative integer such that the element $p^{k_p}x$ is divisible by all powers of p. If such a number m_p or k_p doesn't exist then $m_p = \infty$ or $k_p = \infty$, respectively. The characteristics $char\,(x) = (m_p)$ and $cochar\,(x) = (k_p)$ are called the ***characteristic*** and the ***co-characteristic*** of the element x in the module M. The type $[cochar\,(x)]$ is called the ***co-type*** of the element x. The co-characteristic is an analog of the order of an element. If $x \in Z_\chi$ then $cochar\,(x) = \chi - char\,(x)$ and $char\,(x) \geq \chi - cochar\,(x)$, the inequality can be strict.

The ring $R = \left\langle 1, \bigoplus\limits_p \widehat{Z}_p \right\rangle_* \subset \widehat{Z}$ is called the ring of *pseudo-rational* numbers. See [8] for basic properties of this ring, where the concept has been introduced. The mentioned class of mixed groups $\mathcal{G}$ has the following characterization. The category of groups $\mathcal{G}$ coincides with the category of all finitely generated R-modules such that their p-components are torsion for all prime numbers p (Theorem 5.2 in [8]). A.V. Tsarev is developing an interesting theory of modules over the ring of pseudo-rational numbers in [11,15,16] which is very close to the quotient divisible group theory.

For every characteristic $\chi = (m_p)$ we define the ideal $J_\chi = \bigoplus\limits_p p^{m_p}\widehat{Z}_p$ of the ring R, assuming $p^\infty = 0$, and the ring $R\,(\chi) = R/J_\chi$. An inequality $\chi \geq \kappa$ for two characteristics implies the inclusion $J_\chi \subset J_\kappa$ which determines in turn the natural homomorphism of rings

$$g^\chi_\kappa : R\,(\chi) \to R\,(\kappa) \text{ for } \chi \geq \kappa.$$

In this paper we are interested in the subrings $R^\chi = \langle 1 \rangle_* \subset R\,(\chi)$ of the rings $R\,(\chi)$ keeping the same notation for their additive groups as well. Restrictions of g^χ_κ on R^χ induce the following spectrum of homomorphisms of rings (abelian groups)

$$g^\chi_\kappa : R^\chi \to R^\kappa \text{ for } \chi \geq \kappa. \tag{2.1}$$

Note that the homomorphisms (2.1) are not necessarily surjective, for example, the natural embedding $Z \to Q$ is exactly the homomorphism g^χ_κ for the pair of characteristics $\chi = (\infty, \infty, \ldots) \geq \kappa = (0, 0, \ldots)$. If a characteristic $\chi = (m_p)$ belongs to a nonzero type, then the ring R^χ coincides with the subring $\langle 1 \rangle_* \subset Z_\chi$. In this case, the co-characteristic of 1 in Z_χ coincides with χ. This is one of the reasons why we'll call the characteristic χ as the ***co-characteristic*** of the group R^χ. If a characteristic $\chi = (m_p)$ belongs to the zero type, then $R^\chi = Z_m \oplus Q$, where $m = \prod\limits_p p^{m_p}$.

3 Quotient divisible groups

Definition ([1]) An abelian group A without nonzero torsion divisible subgroups is called *quotient divisible* if it contains a free subgroup F of finite rank such that the quotient group A/F is torsion divisible. Every free basis $x_1, \ldots, x_n$ of the group F is called a *basis* of the quotient divisible group A, the number n is the *rank* of A.$\blacksquare$

The groups R^χ serve as examples of the quotient divisible groups. The rank of R^χ is equal to 1 and a basis of R^χ is the unity element $1 \in R^\chi$ considering R^χ as a ring. And what is more, every quotient divisible group of rank 1 is isomorphic to a group R^χ for some characteristic χ, and $R^\chi \cong R^\kappa \Longleftrightarrow \chi = \kappa$ (see [7]). Therefore an arbitrary quotient divisible group A of rank 1 with a basis x may be denoted as $A = xR^\chi$ and the characteristic χ is the *co-characteristic* of the quotient divisible rank-1 group A.

As it is shown in [1] and [10], every reduced quotient divisible group A can be presented as a pure hull $A \cong \langle x_1, \ldots, x_n \rangle_* \subset M$ of a linearly independent over Z set of elements $x_1, \ldots, x_n$ of a finitely presented $\widehat{Z}$-module M such that $M = \langle x_1, \ldots, x_n \rangle_{\widehat{Z}}$. Namely, $M = \widehat{A}$ is the Z-adic completion of A and the set $x_1, \ldots, x_n$ is the image of a basis of A in $\widehat{A}$.

The divisible part of a quotient divisible group is a divisible torsion free group of finite rank. A reduced complement of the divisible part is not necessarily quotient divisible, for example it is true for a group R^χ, where χ is a nonzero characteristic of the zero type. In general, this complement is a direct sum of a finite group and a quotient divisible reduced group.

The following lemma is useful for us. Let p be a prime number, we understand under p-rank of a group A the dimension of the vector space A/pA over the field Z_p.

Lemma 1 *(of complement) Let A be a quotient divisible group and $\langle t \rangle$ a cyclic p-primary group for a prime number p. If the p-rank of A is strictly less than the rank of A, then the group $A \oplus \langle t \rangle$ is quotient divisible as well.*

Proof Let $x_1, \ldots, x_n$ be a basis of A and $r = rank_p A, r < n$. The vector space A/pA over the field Z_p is generated by the set of vectors $\overline{x}_1 = x_1 + pA, \ldots, \overline{x}_n = x_n + pA$. This set of vectors contains a basis, say $\overline{x}_1, \ldots, \overline{x}_r$, of A/pA. Then the set of elements $x_1, \ldots, x_r, x_{r+1} + t, x_{r+2}, \ldots, x_n$ is a basis of the quotient divisible group $B = A \oplus \langle t \rangle$.$\blacksquare$

4 Duality

The duality [1] between the quotient divisible groups and the torsion free groups of finite rank can be considered as a part of a more general construction as it has been done in [10]. Namely, we have a commutative diagram of the following category functors.

$$\begin{tikzcd} & \mathcal{RM} \arrow[dl, shift left, "c"] \arrow[dr, shift left, "b"] & \\ \mathcal{QD} \arrow[ur, shift left, "c'"] \arrow[rr, shift left, "d"] & & \mathcal{QTF} \arrow[ul, shift left, "b'"] \arrow[ll, shift left, "d'"] \end{tikzcd} \tag{4.1}$$

It is convenient to consider three objects simultaneously. Thus we prefer to call the situation "the triplicity" such that the duality d and d' is a part of it. We briefly introduce now all three categories and the functors referring a reader to [10] for details.

(i) An object of the category $\mathcal{RM}$ is an arbitrary sequence of elements $x_1^0, \ldots, x_n^0$ of an arbitrary finitely presented $\widehat{Z}$-module. Note that, choosing a basis $y_1, \ldots, y_m$ of the module $M = \langle x_1^0, \ldots, x_n^0 \rangle_{\widehat{Z}} = y_1 Z_{\chi_1} \oplus \ldots \oplus y_m Z_{\chi_m}$,we obtain a matrix $\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$ with $\alpha_{ij} \in Z_{\chi_i}$ at the equalities $x_i^0 = \alpha_{1i} y_1 + \ldots + \alpha_{mi} y_m$, $i = 1, \ldots, n$. The matrix is reduced, i.e. its columns generate the $\widehat{Z}$-module M. This point of view has been employed in [10]. That is why the category $\mathcal{RM}$ is called the category of reduced matrices.

(ii) An object of the category $\mathcal{QTF}$ is a pair consisting of a torsion-free finite-rank group A and its basis $x_1, \ldots, x_n$, that is a maximal linearly independent set of elements. For an object $x_1^0, \ldots, x_n^0$ of the category $\mathcal{RM}$, the object $b\left(x_1^0, \ldots, x_n^0\right)$ of the category $\mathcal{QTF}$ is defined in the following way. We define A as a group located between a free group F and a divisible group V

$$F = x_1 Z \oplus \ldots \oplus x_n Z \subset A \subset x_1 Q \oplus \ldots \oplus x_n Q = V.$$

For elements $\gamma_1 = \frac{a_1}{k} + Z, \ldots, \gamma_n = \frac{a_n}{k} + Z$ of the group Q/Z, we define $\gamma_1 x_1 + \ldots + \gamma_n x_n = k^{-1}\left(a_1 x_1 + \ldots + a_n x_n\right) \in V$. Then

$$A = \left\langle f\left(x_1^0\right) x_1 + \ldots + f\left(x_n^0\right) x_n \,\middle|\, f \in Hom_{\widehat{Z}}\left(M, Q/Z\right) \right\rangle,$$

reminding $M = \langle x_1^0, \ldots, x_n^0 \rangle_{\widehat{Z}}$. Conversely (the functor b'), we define a function $x_i^0 : A/F \to Q/Z$ in the following way. Let $z = k^{-1}\left(a_1 x_1 + \ldots + a_n x_n\right) + F \in A/F$. Then $x_i^0(z) = \frac{a_i}{k} + Z \in Q/Z, i = 1, \ldots, n$. Thus the elements $x_1^0, \ldots, x_n^0$ belong to the finitely presented $\widehat{Z}$-module $M = Hom_{\widehat{Z}}\left(A/F, Q/Z\right)$. Note that the group $M = Hom_{\widehat{Z}}\left(A/F, Q/Z\right)$ with the Z-adic topology coincides with the group of Pontryagin's characters ([17]) for the discrete group A/F.

(iii) An object of the category $\mathcal{QD}$ is a pair consisting of a quotient divisible group A^* and its basis $x_1^*, \ldots, x_n^*$. The object $c\left(x_1^0, \ldots, x_n^0\right)$ of the category $\mathcal{QD}$ is defined in the following way. Let $d_1, \ldots, d_n$ be a linearly independent set of elements of a torsion-free divisible group D. Then

$$x_1^* = x_1^0 + d_1, \ldots, x_n^* = x_n^0 + d_n \tag{4.2}$$

is a linearly independent set of the group $M \oplus D$, where M is here the additive group of the $\widehat{Z}$-module $M = \langle x_1^0, \ldots, x_n^0 \rangle_{\widehat{Z}}$. And at last, $A^* = \langle x_1^*, \ldots, x_n^* \rangle_*$ is the pure hull of the elements $x_1^*, \ldots, x_n^*$ in the group $M \oplus D$. It is clear that this definition of the quotient divisible group A^* doesn't depend up to isomorphism on the choice of the elements $d_1, \ldots, d_n$. Nevertheless we can use further the freedom of choice for the elements $d_1, \ldots, d_n$ considering inclusions in the Theorem 3. Conversely (the functor c'), let $\mu : A^* \to \widehat{A^*}$ be the Z-adic completion of a quotient divisible group A^* (see [13], Chapter 39). Then $x_1^0 = \mu(x_1^*), \ldots, x_n^0 = \mu(x_n^*)$.

The functors d and d' are defined like this $d = bc'$ and $d' = cb'$.

The morphisms from an object $(x_1^0, \ldots, x_n^0)$ to an object $(z_1^0, \ldots, z_k^0)$ of the category $\mathcal{RM}$ are pairs (φ, T), where $\varphi : \langle x_1^0, \ldots, x_n^0 \rangle_{\widehat{Z}} \to \langle z_1^0, \ldots, z_k^0 \rangle_{\widehat{Z}}$ is a quasi-homomorphism of the $\widehat{Z}$-modules and T is a matrix with rational entries of dimension $k \times n$, such that the equality $(\varphi x_1^0, \ldots, \varphi x_n^0) = (z_1^0, \ldots, z_k^0)\, T$ takes place in the module $Q \otimes \langle z_1^0, \ldots, z_k^0 \rangle_{\widehat{Z}}$. The morphisms of the categories $\mathcal{QD}$ and $\mathcal{QTF}$ are the quasi-homomorphisms of groups. The functors b and c transform a morphism (φ, T) of the category $\mathcal{RM}$ to the morphisms $f : B \to A$ in $\mathcal{QTF}$ and $f^* : A^* \to B^*$ in $\mathcal{QD}$, where $B, z_1, \ldots, z_k$ and $B^*, z_1^*, \ldots, z_k^*$ correspond to the object $z_1^0, \ldots, z_k^0$, the quasi-homomorphisms f and f^* are defined by the matrix equalities $\begin{pmatrix} f(z_1) \\ \cdots \\ f(z_k) \end{pmatrix} = T \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ and $(f^*(x_1^*), \ldots, f^*(x_n^*)) = (z_1^*, \ldots, z_k^*)\, T$.

It is shown in [10] that the mutually inverse functors c and c' present a category equivalence. The functors b and b' present a category duality, which can be considered as a modern version of the description by Kurosh-Malcev-Derry. The functors d and d' present the category duality [1].

Note that our definitions of the categories $\mathcal{QD}$ and $\mathcal{QTF}$ differ a little bit from the original definitions in [1] and [10], where the objects are groups and the morphisms are quasi-homomorphisms. But evidently our definitions (with fixed bases) give the equivalent categories and we may keep the same notations for them. The basis fixing gives an advantage for the investigations of almost completely decomposable groups. It allows to introduce the following definition.

Definition A *triple* is a set of three objects of the categories $\mathcal{RM}$, $\mathcal{QD}$ and $\mathcal{QTF}$ such that each of them corresponds to each other at the functors of the diagram (4.1). Namely, it is:

- A set of elements $x_1^0, \ldots, x_n^0$ of a finitely presented $\widehat{Z}$-module,
- A torsion-free finite-rank group A with a basis $x_1, \ldots, x_n$,
- A quotient divisible group A^* with a basis $x_1^*, \ldots, x_n^*$. ■

We underline that every element of the triple determines uniquely the remaining two objects of the triple. Moreover, we consolidate further this notation for a triple to simplify formulations without an additional explanation.

Theorem 2 ([10]) *The following statements are equivalent for a triple:*

(i) $A = B \oplus C$, *where* $B = \langle x_1, \ldots, x_k \rangle_*$ *and* $C = \langle x_{k+1}, \ldots, x_n \rangle_*$,

(ii) $\langle x_1^0, \ldots, x_n^0 \rangle_{\widehat{Z}} = \langle x_1^0, \ldots, x_k^0 \rangle_{\widehat{Z}} \oplus \langle x_{k+1}^0, \ldots, x_n^0 \rangle_{\widehat{Z}}$,

(iii) $A^* = B^* \oplus C^*$. ■

In this theorem, the groups B, B^* and the set $x_1^0, \ldots, x_k^0$ form a separate triple as well as the groups C, C^* with the set $x_{k+1}^0, \ldots, x_n^0$. In particular, we use the pure hull $\left\langle x_1^0, \ldots, x_k^0 \right\rangle_*$ in $\left\langle x_1^0, \ldots, x_k^0 \right\rangle_{\widehat{Z}}$, but not in $\left\langle x_1^0, \ldots, x_n^0 \right\rangle_{\widehat{Z}}$, by the construction of the reduced part of the group B^* at the functor c.

5 Change of bases

Two different bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of a torsion free group A give us two different triples. One of them contains also a set $x_1^0, \ldots, x_n^0$ of a finitely presented $\widehat{Z}$-module and a quotient divisible group A_X^* with a basis $x_1^*, \ldots, x_n^*$. The second one contains a set $y_1^0, \ldots, y_n^0$ and a quotient divisible group A_Y^* with a basis $y_1^*, \ldots, y_n^*$. The following theorem considers relations between them.

Theorem 3 *Let two bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of a torsion free group A be written in the form of columns X and Y. If $X = SY$ for a nonsingular matrix S with integer entries, then:*

(i) *The $\widehat{Z}$-module $\left\langle y_1^0, \ldots, y_n^0 \right\rangle_{\widehat{Z}}$ is a submodule of index $|\det S|$ of the module $\left\langle x_1^0, \ldots, x_n^0 \right\rangle_{\widehat{Z}}$ and the following matrix equality takes place*

$$\left(y_1^0, \ldots, y_n^0\right) = \left(x_1^0, \ldots, x_n^0\right) S. \tag{5.1}$$

(ii) *Defining the bases of the groups A_X^* and A_Y^* by the equalities (4.2), we can choose elements $d_1, \ldots, d_n$ in these equalities such that $A_Y^* \subset A_X^*$ and the following equality takes place*

$$(y_1^*, \ldots, y_n^*) = (x_1^*, \ldots, x_n^*)\, S. \tag{5.2}$$

Moreover $|A_X^*/A_Y^*| = |\det S|$, where $|\det S|$ is the absolute value of the determinant.

Proof Since $X = SY$, the following inclusion takes place $F = \langle x_1, \ldots, x_n \rangle \subset G = \langle y_1, \ldots, y_n \rangle \subset A$. Applying the functor $Hom\,(-, Q/Z)$ to the short exact sequence $0 \to G/F \xrightarrow{i} A/F \xrightarrow{j} A/G \to 0$, we obtain the exact sequence of $\widehat{Z}$-modules $0 \to M_1 \xrightarrow{j^*} M \xrightarrow{i^*} Hom\,(G/F, Q/Z) \to 0$, where $M_1 = \left\langle y_1^0, \ldots, y_n^0 \right\rangle_{\widehat{Z}}, M = \left\langle x_1^0, \ldots, x_n^0 \right\rangle_{\widehat{Z}}$ and the $\widehat{Z}$-module on the right is a finite abelian group which is isomorphic to the group G/F.

An arbitrary element z of the group A/F is of the form $z = m^{-1}\left(\sum\limits_{i=1}^{n} a_i x_i\right) + F$, where $a_i \in Z, 0 \neq m \in Z$. Substituting $x_i = \sum\limits_{k=1}^{n} s_{ik} y_k$, where $S = \|s_{ik}\|$ and $X = SY$, we obtain $z = m^{-1}\left(\sum\limits_{i=1}^{n} a_i \sum\limits_{k=1}^{n} s_{ik} y_k\right) + F = m^{-1}\left(\sum\limits_{k=1}^{n}\left(\sum\limits_{i=1}^{n} a_i s_{ik}\right) y_k\right) + F$. By the definition of the elements x_i^0 and y_i^0 in Section 4 "Duality", item 2, the function $z \longmapsto m^{-1}\left(\sum\limits_{i=1}^{n} a_i s_{ik}\right) + Z \in Q/Z$ is exactly the function $j^*\left(y_k^0\right) : A/F \to Q/Z$

and $x_i^0(z) = m^{-1}a_i + Z \in Q/Z$. Identifying $j^*\left(y_k^0\right) = y_k^0$, we obtain finally $y_k^0(z) = m^{-1}\left(\sum_{i=1}^{n} a_i s_{ik}\right) + Z = \sum_{i=1}^{n}\left(m^{-1}a_i + Z\right)s_{ik} = \sum_{i=1}^{n} x_i^0(z)\, s_{ik}$. Since the values of two functions $y_k^0(z)$ and $\sum_{i=1}^{n} x_i^0(z)\, s_{ik}$ coincide for every $z \in A/F$, the functions coincide as well and the equality (5.1) is proved.

The index of M_1 in M is equal to $|G/F|$. The matrix S can be presented in the form $S = T_1 T T_2$, where T_1 and T_2 are invertible, T is diagonal and they are all with integer entries. The matrix equality $X = SY = (T_1 T T_2)\, Y$ implies the equality $T_1^{-1}X = T(T_2 Y)$. Thus the basis $T_1^{-1}X$ of the free group F is expressed over the basis $T_2 Y$ of the free group G with help of the diagonal matrix $T = \begin{pmatrix} t_1 & 0\cdots & 0 \\ \ldots & \ddots & \ldots \\ 0 & 0\cdots & t_n \end{pmatrix}$. Hence $G/F = C_1 \oplus \ldots \oplus C_n$, where the direct summands $C_1, \ldots, C_n$ are cyclic of order $|t_1|, \ldots, |t_n|$, respectively. Therefore $|G/F| = |t_1| \cdot \ldots \cdot |t_n| = |\det S|$. It accomplishes the proof of the first part of the theorem.

If we just have, say, $x_1^* = x_1^0 + d_1, \ldots, x_n^* = x_n^0 + d_n$, then we choose elements $d_1', \ldots, d_n'$ of the divisible torsion free group $D = \langle d_1, \ldots, d_n \rangle_*$ in such a way that $(d_1', \ldots, d_n') = (d_1, \ldots, d_n)\, S$. Defining $y_1^* = y_1^0 + d_1', \ldots, y_n^* = y_n^0 + d_n'$, we obtain immediately $A_Y^* \subset A_X^*$ and the equality (5.2). The only thing to do is to prove $|A_X^*/A_Y^*| = |\det S|$. Without loss of generality assume $A_X^* = \left\langle x_1^0, \ldots, x_n^0 \right\rangle_*$, that is the set $x_1^0, \ldots, x_n^0$ is linearly independent over Z. The natural homomorphism $\theta : A_X^* \to M/M_1$ is surjective, because the images of elements $x_1^0, \ldots, x_n^0$ generate the finite group M/M_1. The kernel of θ is equal to $A_X^* \cap M_1$ and the intersection $A_X^* \cap \langle y_1^0, \ldots, y_n^0 \rangle_{\widehat{Z}}$ coincides in turn with A_Y^*. We obtain finally $A_X^*/A_Y^* \cong M/M_1 \cong G/F$.■

We distinguish a particular case of Theorem 3.

Corollary 4 *Let $x_1 = my_1, \ldots, x_n = my_n$ for an integer $m \neq 0$. Then $\left\langle y_1^0, \ldots, y_n^0 \right\rangle_{\widehat{Z}} \subset \left\langle x_1^0, \ldots, x_n^0 \right\rangle_{\widehat{Z}}$ and $y_1^0 = mx_1^0, \ldots, y_n^0 = mx_n^0$.*■

Corollary 5 *Let two sequences $x_1^0, \ldots, x_n^0$ and $y_1^0, \ldots, y_n^0$ of elements be given in a finitely presented $\widehat{Z}$-module. If a matrix equality $\left(y_1^0, \ldots, y_n^0\right) = \left(x_1^0, \ldots, x_n^0\right) S$ takes place for an integer matrix S with det$S = \pm 1$, then the torsion free groups coincide and the quotient divisible groups coincide in two triples corresponding to the given sequences. In particular, the groups A and A^* do not depend on the order of elements in the sequence $x_1^0, \ldots, x_n^0$ of a triple.*■

Corollary 6 *The dual quotient divisible (torsion free) group with respect to a basis $x_1, \ldots, x_n$ $(x_1^*, \ldots, x_n^*)$ doesn't depend on the choice of the basis in the free group $F = \langle x_1, \ldots, x_n \rangle$ $(F^* = \langle x_1^*, \ldots, x_n^* \rangle)$. Therefore, it depends only on the choice of the free subgroup F (F^*). Moreover, it doesn't depend even on the choice of the free subgroup up to quasi-equality.*■

The following example shows that an indecomposable quotient divisible group can be dual to a completely decomposable torsion free group.

Example 1 We consider a torsion free group $A = x_1 Q_2 \oplus x_2 Q_5$ with the basis

x_1, x_2. The dual quotient divisible group A^*_X with respect to this basis is of the form $A^*_X = x^*_1 Q^{(2)} \oplus x^*_2 Q^{(5)}$. Let us consider now two new bases of the group A : $y_1 = \frac{1}{3}(x_1 - x_2), y_2 = x_2$ and $z_1 = \frac{1}{3}x_1, z_2 = \frac{1}{3}x_2$. We have $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ in the group A. By Theorem 3 and Corollary 4, we obtain the inclusions for dual quotient divisible groups $A^*_Z \subset A^*_Y \subset A^*_X$ and the relations $z^*_1 = 3x^*_1, z^*_2 = 3x^*_2$ and $z^*_1 = y^*_1, z^*_2 = -y^*_1 + 3y^*_2$ and $y^*_1 = 3x^*_1, y^*_2 = x^*_1 + x^*_2$. Note that $A^*_Z = z^*_1 Q^{(2)} \oplus z^*_2 Q^{(5)} = 3\ A^*_X \cong A^*_X$ and $A^*_Y = \left\langle A^*_Z, \frac{z^*_1 + z^*_2}{3} \right\rangle$. The group A^*_Y is indecomposable (see [13], Example 88.2).■

6 Almost completely decomposable groups

For a characteristic χ, we denote by R_χ the subgroup of Q such that $1 \in R_\chi$ and the characteristic of 1 in R_χ is equal to χ. The lattice of characteristics gives a spectrum of the natural embeddings

$$f^\chi_\kappa : R_\kappa \to R_\chi \text{ for } \kappa \leq \chi, \tag{6.1}$$

where $f^\chi_\kappa(1) = 1$. The quotient divisible group R^χ is dual to R_χ with respect to the natural basis $1 \in R_\chi$ and the dual basis is $1^* = 1 \in R^\chi$. The homomorphisms (2.1) g^χ_κ are dual to f^χ_κ and the following spectrum of the homomorphisms of the quotient divisible groups is dual to (6.1)

$$g^\chi_\kappa : R^\chi \to R^\kappa \text{ for } \kappa \leq \chi, \tag{6.2}$$

where $g^\chi_\kappa(1) = 1$. It is interesting to note that every group of the spectrum (6.2) is naturally a ring and then g^χ_κ are homomorphisms of rings, while the groups of the spectrum (6.1) are subrings of Q if and only if they are quotient divisible. $R^\chi = R_\kappa \Longleftrightarrow \chi \vee \kappa = (\infty, \infty, \ldots)$ and $\chi \wedge \kappa = (0, 0, \ldots)$.

An arbitrary torsion-free rank-1 group is of the form $A = xR_\chi$, where x is its basis and χ is the characteristic of x. The dual to A quotient divisible group is $A^* = x^* R^\chi$. The last group can be considered sometimes as a free rank-1 module over the ring R^χ. The bases x and x^* are mutually dual. The following theorem describes triples for the completely decomposable groups.

Theorem 7 ([10]) *The following statements are equivalent for a triple:*

(i) *The set of elements $x^0_1, \ldots, x^0_n$ is linearly independent over $\widehat{Z}$ and the co-characteristics of these elements are $\chi_1, \ldots, \chi_n$, respectively.*

(ii) $A^0 = x^0_1 Z_{\chi_1} \oplus \ldots \oplus x^0_n Z_{\chi_n}$, *where* $A^0 = \langle x^0_1, \ldots, x^0_n \rangle_{\widehat{Z}}$.

(iii) $A = x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n}$.

(iv) $A^* = x^*_1 R^{\chi_1} \oplus \ldots \oplus x^*_n R^{\chi_n}$.■

We are generalizing this theorem on the almost completely decomposable groups in the present section.

Definition A set of elements $y_1, \ldots, y_n$ of a finitely presented $\widehat{Z}$-module is called *almost linearly independent over* $\widehat{Z}$ if the equality $\alpha_1 y_1 + \ldots + \alpha_n y_n = 0$ with universal integer coefficients implies that all the elements $\alpha_1 y_1, \ldots, \alpha_n y_n$ have finite order, that is $m\alpha_1 y_1 = \ldots = m\alpha_n y_n = 0$ for some non-zero integer m.■

Lemma 8 *Let $y_1, \ldots, y_n$ be an almost linearly independent set of elements of a finitely presented $\widehat{Z}$-module $M = \langle y_1, \ldots, y_n \rangle_{\widehat{Z}}$. Then there exist elements of finite order $t_1, \ldots, t_n \in M$ such that $M = \langle y_1 + t_1, \ldots, y_n + t_n \rangle_{\widehat{Z}}$ and the set of the elements $y_1 + t_1, \ldots, y_n + t_n$ is linearly independent over $\widehat{Z}$.*

Proof For every $i = 1, \ldots, n$, the $\widehat{Z}$-module $T_i = \langle y_i \rangle_{\widehat{Z}} \cap \langle y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n \rangle_{\widehat{Z}}$ is finitely presented and cyclic, that is it is isomorphic to $\widehat{Z}_\chi$ for a characteristic χ. Since the set $y_1, \ldots, y_n$ is almost linearly independent, it follows that χ belongs to the zero type and hence T_i is a cyclic group. Consider the set $P = \{p_1, \ldots, p_s\}$ of prime divisors of the orders of the groups $T_1, \ldots, T_n$ and carry out the following operation for a prime number $p \in P$.

We remind that as every finitely presented $\widehat{Z}$-module the module M is of the form $M = \prod\limits_p M_p$, where $M_p = \langle a_1 \rangle_{\widehat{Z}_p} \oplus \ldots \oplus \langle a_n \rangle_{\widehat{Z}_p}$. The first r direct summands are p-primary cyclic groups and the remaining summands are isomorphic to $\widehat{Z}_p, 0 \le r \le n$. We obtain a direct decomposition $M = \langle a_1 \rangle_{\widehat{Z}} \oplus \ldots \oplus \langle a_r \rangle_{\widehat{Z}} \oplus N$, where the $\widehat{Z}$-module N has no p-torsion, and also we obtain the equalities $y_1 = s_1 + y_1', \ldots, y_n = s_n + y_n'$ with respect to this decomposition, where $s_1, \ldots, s_n \in \langle a_1 \rangle_{\widehat{Z}} \oplus \ldots \oplus \langle a_r \rangle_{\widehat{Z}}$ and $y_1', \ldots, y_n' \in N$. Let ε_p be the universal integer such that all its components are zeros except for the p-component which is equal to 1. The elements $\varepsilon_p y_1', \ldots, \varepsilon_p y_n'$ generate the free p-adic module $N_p = \varepsilon_p N$ of rank $n - r$. Exactly $n - r$ elements in the sequence $\varepsilon_p y_1', \ldots, \varepsilon_p y_n'$ are different from 0, otherwise we obtain a contradiction with the property of the almost linear independence. Therefore, r elements, say $\varepsilon_p y_1', \ldots, \varepsilon_p y_r'$, are equal to 0. We define now $z_1 = a_1 + y_1', \ldots, z_r = a_r + y_r', z_{r+1} = y_{r+1}', \ldots, z_n = y_n'$. It is easy to see that $z_1 = y_1 + t_1, \ldots, z_n = y_n + t_n$, where $t_1, \ldots, t_n$ are p-primary elements of the module $M, M = \langle z_1, \ldots, z_n \rangle_{\widehat{Z}}$, and the set of elements $z_1 = y_1 + t_1, \ldots, z_n = y_n + t_n$ is almost linearly independent.

The corresponding set of prime numbers for the almost linearly independent set of elements $z_1, \ldots, z_n$ is equal to $P \setminus \{p\}$. By the hypothesis of induction the statement of lemma takes place for the elements $z_1, \ldots, z_n$, therefore it takes place for the set $y_1, \ldots, y_n$ as well.■

Theorem 9 *The following statements are equivalent for a triple.*

(i) *The group A contains a subgroup of finite index of the form $x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n}$ for some characteristics $\chi_1, \ldots, \chi_n$.*

(ii) *The set of elements $x_1^0, \ldots, x_n^0$ is almost linearly independent over $\widehat{Z}$.*

(iii) *There exist torsion elements $t_1, \ldots, t_n \in A^*$ such that $A^* = (x_1^* + t_1) R^{\kappa_1} \oplus \ldots \oplus (x_n^* + t_n) R^{\kappa_n}$. Moreover, $[\kappa_1] = [\chi_1], \ldots, [\kappa_n] = [\chi_n]$.*

Proof $1 \to 2$. Denote $B = x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n}$ and $F = \langle x_1, \ldots, x_n \rangle$. The exact sequence $0 \to B \to A \to C \to 0$ induces the exact sequence $0 \to B/F \to A/F \to C \to 0$ with a finite group $C \cong A/B$. Applying the functor $Hom(-, Q/Z)$ to the last sequence, we obtain an exact sequence $0 \to C^0 \to A^0 \to B^0 \to 0$ of $\widehat{Z}$-module homomorphisms, where the group $C^0 = Hom\,(C, Q/Z) \cong C$ is finite. The elements $x_1^0, \ldots, x_n^0$ of the triple are located in the $\widehat{Z}$-module A^0. Suppose $\alpha_1 x_1^0 + \ldots + \alpha_n x_n^0 = 0$ with universal integer coefficients. Passing to the $\widehat{Z}$-module $B^0 = x_1^0 Z_{\chi_1} \oplus \ldots \oplus x_n^0 Z_{\chi_n}$ we obtain the equalities $\alpha_1 x_1^0 = \ldots = \alpha_n x_n^0 = 0$ in B^0. Therefore, the elements $\alpha_1 x_1^0, \ldots, \alpha_n x_n^0$ belong to the image of C^0 in A^0, that is they are periodic. Hence the set of elements $x_1^0, \ldots, x_n^0 \in A^0$ is almost linearly independent over $\widehat{Z}$.

$2 \to 3$. Let the set $x_1^0, \ldots, x_n^0$ be almost linearly independent over $\widehat{Z}$. By Lemma 8, $A^0 = \left\langle x_1^0, \ldots, x_n^0 \right\rangle_{\widehat{Z}} = \left\langle x_1^0 + t_1, \ldots, x_n^0 + t_n \right\rangle_{\widehat{Z}} = \left\langle x_1^0 + t_1 \right\rangle_{\widehat{Z}} \oplus \ldots \oplus \left\langle x_n^0 + t_n \right\rangle_{\widehat{Z}}$ for some torsion elements $t_1, \ldots, t_n \in A^0$. Since the torsion parts of the groups A^0 and A^* coincide, $t_1, \ldots, t_n \in A^*$. Moreover, it is easy to see that $\left\langle x_1^0, \ldots, x_n^0 \right\rangle_* = \left\langle x_1^0 + t_1, \ldots, x_n^0 + t_n \right\rangle_* = \left\langle x_1^0 + t_1 \right\rangle_* \oplus \ldots \oplus \left\langle x_n^0 + t_n \right\rangle_*$. The last pure hulls are considered in the modules $\left\langle x_1^0 + t_1 \right\rangle_{\widehat{Z}}, \ldots, \left\langle x_n^0 + t_n \right\rangle_{\widehat{Z}}$, respectively. Thus we obtain $A^* = (x_1^* + t_1)\, R^{\kappa_1} \oplus \ldots \oplus (x_n^* + t_n)\, R^{\kappa_n}$, where $\kappa_1, \ldots, \kappa_n$ are the co-characteristics of the elements $x_1^0 + t_1, \ldots, x_n^0 + t_n$ in the module A^0, which are equivalent to the co-characteristics of the elements $x_1^0, \ldots, x_n^0$ in the module A^0, which are equivalent in turn to the characteristics $\chi_1, \ldots, \chi_n$, respectively.

$3 \to 1$. We have two bases in the quotient divisible group A^*, namely $x_1^*, \ldots, x_n^*$ and $y_1^* = x_1^* + t_1, \ldots, y_n^* = x_n^* + t_n$. The dual torsion free group with respect to the first basis coincides with the group A of the given triple, the fixed basis of A is $x_1, \ldots, x_n$. The dual group with respect to the second basis $y_1^*, \ldots, y_n^*$ belongs to other triple. We denote it as A_Y, its basis $y_1, \ldots, y_n$ is dual to the basis $y_1^*, \ldots, y_n^* \in A^*$. By Theorem 7, $A_Y = y_1 R_{\kappa_1} \oplus \ldots \oplus y_n R_{\kappa_n}$, where $\kappa_1, \ldots, \kappa_n$ are co-characteristics of the elements $x_1^0 + t_1, \ldots, x_n^0 + t_n$ in the module A^0.

There exists a non-zero integer m such that $mx_1^* = my_1^*, \ldots, mx_n^* = my_n^*$. The homomorphism $f : A^* \to A^*$ with $f\,(z) = mz$ induces two dual quasi-homomorphisms $f_1^* : A \to A_Y$ and $f_2^* : A_Y \to A$ according two different triples. By the definitions of Section 4 "Duality", $f_1^*\,(x_1) = my_1, \ldots, f_1^*\,(x_n) = my_n$ and $f_2^*\,(y_1) = mx_1, \ldots, f_2^*\,(y_n) = mx_n$. For a non-zero integer k, two morphisms kf_1 and kf_2 are not only homomorphisms, but monomorphisms as well. Identifying along the monomorphisms kf_1 and kf_2, we obtain the inclusions

$$\left(k^2 m^2 y_1\right) R_{\kappa_1} \oplus \ldots \oplus \left(k^2 m^2 y_n\right) R_{\kappa_n} \subset A \subset y_1 R_{\kappa_1} \oplus \ldots \oplus y_n R_{\kappa_n}.$$

Since $kmy_i = x_i, i = 1, ..., n$, under the identification, it follows that the first inclusion is of the form $(kmx_1)\, R_{\kappa_1} \oplus \ldots \oplus (kmx_n)\, R_{\kappa_n} \subset A$. Thus we obtain $(kmx_1)\, R_{\kappa_1} \oplus \ldots \oplus (kmx_n)\, R_{\kappa_n} \subset x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n} \subset A$. The index of the subgroup is not greater than $(mk)^{2n}$, and the characteristics $\chi_1, \ldots, \chi_n$ are equivalent to the characteristics $\kappa_1, \ldots, \kappa_n$, respectively. ■

Example 1 shows in particular that the quotient divisible group dual to an almost completely decomposable group is not necessarily decomposed into a direct sum of subgroups. But nevertheless the following corollary of Theorem 9 takes place.

Corollary 10 *Every almost completely decomposable group contains a basis such that the dual quotient divisible group with respect to it is decomposed into a direct sum* *of quotient divisible groups of rank* 1.■

7 Dualization of a Lemma by L.Fuchs

Lemma by L.Fuchs [18] gives a sufficient condition of indecomposability for a torsion-free finite-rank group, see Lemma 88.3 in [13]. The following theorem is a dualization of this lemma.

Theorem 11 *Let a set $x_1^0, \ldots, x_n^0$ of elements of a finitely presented $\widehat{Z}$-module determine a triple with a torsion free group A. If:*

(i) *The set $x_1^0, \ldots, x_n^0$ is almost linearly independent,*

(ii) *The co-characteristics $\chi_1, \ldots, \chi_n$ of $x_1^0, \ldots, x_n^0$ belong to pairwise incomparable types,*

(iii) $\langle x_1^0 \rangle_{\widehat{Z}} \cap \langle x_i^0 \rangle_{\widehat{Z}} \neq 0$ *for each $i = 2, \ldots, n$,*

then the group A is not decomposable into a direct sum of nonzero subgroups.

Proof We show first that the set $x_1^0, \ldots, x_n^0$ is linearly independent over Z. Let $m_1 x_1^0 + \ldots + m_n x_n^0 = 0$, $m_1, \ldots, m_n \in Z$. If, say, $m_1 \neq 0$, then the element $m_1 x_1^0$ is periodic because of the first condition. Therefore $[\chi_1] = 0$ and this is a contradiction with the second condition. Thus $m_1 = \ldots = m_n = 0$. By Section 4 "Duality", we obtain that $x_1^* = x_1^0, \ldots, x_n^* = x_n^0$ and $A^* = \langle x_1^0, \ldots, x_n^0 \rangle_*$.

Let $x \in A^*$ be an arbitrary element of infinite order and χ be its co-characteristic in $\langle x_1^0, \ldots, x_n^0 \rangle_{\widehat{Z}}$. Then $mx = m_1 x_1^0 + \ldots + m_n x_n^0$ for some integer coefficients with $m \neq 0$. Multiplying the equality by an arbitrary universal number α of characteristic χ, we obtain $m_1 \alpha x_1^0 + \ldots + m_n \alpha x_n^0 = 0$. Since all the summands must be periodic, we obtain that $[\chi] \geq [\chi_i]$ for every i with $m_i \neq 0$. Thus the co-type of x is greater than or equal to at least one of the co-types of elements $x_1^0, \ldots, x_n^0$. Suppose now that $[\chi] \leq [\chi_j]$ for some j, then $[\chi_i] \leq [\chi] \leq [\chi_j]$, and by the second condition we obtain $i = j$ and $[\chi] = [\chi_j]$. In this case, only one coefficient m_j is different from zero in the equality $mx = m_1 x_1^0 + \ldots + m_n x_n^0$ on the right. Hence $mx = m_j x_j^0$ for some non-zero integers m and m_j. The torsion elements of A^* have the zero co-type. Thus it is proved that if $cotype\,(x) \leq cotype\,(x_i^0)$ for $x \in A^*$ and some $i = 1, \ldots, n$, then the elements x and x_i^0 are colinear or x is torsion.

Let us suppose now that the torsion free group A with the basis $x_1, \ldots, x_n$ is decomposed into a direct sum of non-zero subgroups. Then there exists a basis $y_1, \ldots, y_n \in \langle x_1, \ldots, x_n \rangle \subset A$ such that $\langle y_1, \ldots, y_k \rangle_* \oplus \langle y_{k+1}, \ldots, y_n \rangle_* = A, 0 < k < n$. Moreover $\begin{pmatrix} y_1 \\ \cdots \\ y_n \end{pmatrix} = S \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$, where S is a nonsingular matrix with integer entries. Applying Theorem 3, we obtain that $A^* = A_X^* \subset A_Y^*$ and $(x_1^*, \ldots, x_n^*) = (y_1^*, \ldots, y_n^*)\, S$. By Theorem 2, $A_Y^* = B \oplus C$, where $B = \langle y_1^0, \ldots, y_k^0 \rangle_*$ in $\langle y_1^0, \ldots, y_k^0 \rangle_{\widehat{Z}}$ and $C = \langle y_{k+1}^0, \ldots, y_n^0 \rangle_*$ in $\langle y_{k+1}^0, \ldots, y_n^0 \rangle_{\widehat{Z}}$. By the projections $A_Y^* \to B$ and $A_Y^* \to C$, the co-characteristics of elements are decreasing as it takes place for any homomorphism of quotient divisible groups. Since A^* and A_Y^* are quasi-equal, the sets of their co-types coincide. Therefore one of the projections of the element x_i^* must have the co-type $[\chi_i]$ and the other projection has the co-type 0 for every $i = 1, \dots, n$. Thus $mx_i^* \in B$ or $mx_i^* \in C$ for a suitable integer $m \neq 0$. On the other hand, $x_i^0 = s_{1i}y_1^0 + \dots + s_{ni}y_n^0$, where s_{ki} are the entries of the matrix S. If $mx_i^0 \in B$, then necessarily $s_{k+1i} = \dots = s_{ni} = 0$, hence $x_i^* = x_i^0 = s_{1i}y_1^0 + \dots + s_{ki}y_k^0 \in B$. We obtain $x_i^* = x_i^0 \in B$ or $x_i^* = x_i^0 \in C$ for every $i = 1, ..., n$. Let $x_1^0 \in B$ and $x_j^0 \in C$ for some j. Then for some element $0 \neq t \in \langle x_1^0 \rangle_{\widehat{Z}} \cap \langle x_j^0 \rangle_{\widehat{Z}}$, we obtain $t \in B \cap C$ and it is a contradiction. Thus the group A is indecomposable. ■

8 Completely decomposable homogeneous groups

Theorem 12 *Let $y_1^0, \dots, y_n^0$ be a linearly independent over $\widehat{Z}$ set of elements of a finitely presented $\widehat{Z}$-module, such that the co-characteristics $\chi_1, \dots, \chi_n$ of $y_1^0, \dots, y_n^0$ are equal $\chi_1 = \dots = \chi_n = \chi$. Let a set of elements $x_1^0, \dots, x_n^0$ of the same module be defined by a matrix equality $\left(x_1^0, \dots, x_n^0\right) = \left(y_1^0, \dots, y_n^0\right) S$, where S is an integer matrix of dimension $n \times n$ with det$S = \pm 1$. Then the triple corresponding to the set $x_1^0, \dots, x_n^0$ has the following properties:*

(i) *The set $x_1^0, \dots, x_n^0$ is linearly independent over $\widehat{Z}$ and all the co-characteristics of the elements are equal to χ,*

(ii) $A = x_1 R_\chi \oplus \dots \oplus x_n R_\chi$,

(iii) $A^* = x_1^* R^\chi \oplus \dots \oplus x_n^* R^\chi$.

Proof The module $M = \left\langle y_1^0, \dots, y_n^0 \right\rangle_{\widehat{Z}} = y_1^0 Z_\chi \oplus \dots \oplus y_n^0 Z_\chi$ is a free module over the ring Z_χ as well. The correspondence $y_1^0 \longmapsto x_1^0, \dots, y_n^0 \longmapsto x_n^0$ determines an automorphism of the Z_χ-module M which maps the free basis $y_1^0, \dots, y_n^0$ to the free basis $x_1^0, \dots, x_n^0$. It proves the first statement of the theorem. Applying Theorem 7, we finish the proof. ■

Corollary 13 *Let a quotient divisible group $B = y_1 R^\chi \oplus \dots \oplus y_n R^\chi$ be a direct sum of copies isomorphic to R^χ. For every integer matrix S of dimension $n \times n$ with det$S = \pm 1$, the set $x_1, \dots, x_n \in B$, defined by the matrix equality $(x_1, \dots, x_n) = (y_1, \dots, y_n) S$, is a basis of the quotient divisible group B as well. The dual to B torsion free group B^* is the same considering it with respect to each of two bases. Moreover, the following two decompositions of B^* take place with respect to the dual bases: $B^* = y_1^* R_\chi \oplus \dots \oplus y_n^* R_\chi = x_1^* R_\chi \oplus \dots \oplus x_n^* R_\chi$.*■

Theorem 12 would not be true if we replace the equality of the co-characteristics $\chi_1 = \dots = \chi_n = \chi$ by the equivalence of the co-characteristics $\chi_1 \sim \dots \sim \chi_n \sim \chi$. It is shown in the following example.

Example 2 First we define a triple. We consider three characteristics $\chi_1 = (0, 1, 0, 0, ...)$, $\chi_2 = (1, 0, 0, 0, ...)$, $\chi_3 = \chi_1 + \chi_2 = (1, 1, 0, 0, ...)$ and a finitely presented $\widehat{Z}$-module $Z_6 = \left\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\right\}$. Let $y_1^0 = \overline{2}$ and $y_2^0 = \overline{3}$. Then $cochar\left(y_1^0\right) = \chi_1$ and $cochar\left(y_2^0\right) = \chi_2$. According to Section 4 “Duality”, $y_1^* = \overline{2} + d_1, y_2^* = \overline{3} + d_2, y_1^* R^{\chi_1} = \left\langle \overline{2} \right\rangle \oplus d_1 Q, y_2^* R^{\chi_2} = \left\langle \overline{3} \right\rangle \oplus d_2 Q$. Since the set y_1^0, y_2^0 is linearly independent over $\widehat{Z}$, we have the following direct decompositions according to Theorem 7:

(i) $A^* = y_1^* R^{\chi_1} \oplus y_2^* R^{\chi_2} = d_1 Q \oplus d_2 Q \oplus Z_6$,

(ii) $A = y_1 R_{\chi_1} \oplus y_2 R_{\chi_2}$. The rank-1 group $y_1 R_{\chi_1}$ contains an element $v_1 = \frac{1}{3} y_1$ and $y_1 R_{\chi_1} = \langle v_1 \rangle$. Analogously, $y_2 R_{\chi_2} = \langle v_2 \rangle$, where $v_2 = \frac{1}{2} y_2$. Thus $A = v_1 Z \oplus v_2 Z$ is a free group of rank 2 and the fixed basis is $y_1 = 3v_1, y_2 = 2v_2$.

We consider now another triple which is corresponding to the set x_1^0, x_2^0 defined by the matrix equality $\left(x_1^0, x_2^0\right) = \left(y_1^0, y_2^0\right) \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. We note immediately that the groups A and A^* of the new triple are the same, because the matrix is invertible. Only the pair of mutually dual bases is different. Namely, we have $x_1^0 = \overline{2}$, $x_2^0 = \overline{1}$, the co-characteristics of x_1^0 and x_2^0 are χ_1 and χ_3, respectively. Theorem 7 can not be used, because the set x_1^0, x_2^0 is not linearly independent over $\widehat{Z}$, it is only almost linearly independent over $\widehat{Z}$. And we can not obtain a direct decomposition "along" the bases x_1, x_2 and x_1^*, x_2^*. According to Theorem 3, $x_1^* = \overline{2} + (d_1 + 2d_2)$, $x_2^* = \overline{1} + (2d_1 + 3d_2)$ and $x_1 = -9v_1 + 4v_2, x_2 = 6v_1 - 2v_2$. The quotient divisible group A^* contains the quotient divisible rank-1 subgroups $x_1^* R^{\chi_1} = \left\langle \overline{2} \right\rangle + (d_1 + 2d_2) Q$ and $x_2^* R^{\chi_3} = \left\langle \overline{1} \right\rangle + (2d_1 + 3d_2) Q$. Moreover, $A^* = x_1^* R^{\chi_1} + x_2^* R^{\chi_3}$, but $x_1^* R^{\chi_1} \cap x_2^* R^{\chi_3} = \left\langle \overline{2} \right\rangle$ and the sum is not direct. On the other hand, $\langle x_1 \rangle_* = x_1 Z \subset A, \langle x_2 \rangle_* = x_2 R_{\chi_2} \subset A$. Of course, $\langle x_1 \rangle_* \cap \langle x_2 \rangle_* = 0$, but the direct sum $\langle x_1 \rangle_* \oplus \langle x_2 \rangle_*$ doesn't coincide with the group A, it is of the index 3.■

9 Lattice of admissible almost completely decomposable groups

Definition. An element t of an arbitrary group is called *admissible* with respect to a characteristic χ if it is torsion and the p-component of the characteristic χ is equal to zero for every prime divisor p of the order of the element t.■

The next proposition follows easily from Lemma 1.

Proposition 14 *Let xR^{χ} be a rank-1 quotient divisible group of a co-characteristic χ with a basis x and $\langle t \rangle$ be a cyclic group. The group $xR^{\chi} \oplus \langle t \rangle$ is quotient divisible if and only if the element t is admissible with respect to the characteristic χ. Moreover, if $xR^{\chi} \oplus \langle t \rangle$ is quotient divisible, then its rank is* 1 *and the element $x + t$ is its basis.* ■

We fix now an arbitrary sequence of characteristics $\Xi = (\chi_1, \ldots, \chi_n)$ and a basis $x_1, \ldots, x_n$ of a vector space V over Q. The group $B = x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n} \subset V$ is completely decomposable torsion free. The group $B^* = x_1^* R^{\chi_1} \oplus \ldots \oplus x_n^* R^{\chi_n}$ is dual to B quotient divisible. In this section, we consider some finite extensions A of the group B with the same common fixed basis $x_1, \ldots, x_n$ for all them. Every such group A determines a pair: the dual quotient divisible group A^* and the dual basis (to the fixed basis $x_1, \ldots, x_n$). For different groups A those dual bases are different, the dual groups A^* are different of course as well, though they all are quasi-equal. Thus we obtain a fan of different quotient divisible groups and their bases. Connections between them are described in Theorem 15. The different dual bases differ by torsion elements. So the sequences of the torsion elements are terms of this description.

Definition. A sequence of elements $T = (t_1, \ldots, t_n)$ of a group $G_T = \langle t_1, \ldots, t_n \rangle$ is called *admissible* with respect to the sequence of characteristics $\Xi = (\chi_1, \ldots, \chi_n)$ if each element t_i is admissible with respect to the characteristic $\chi_i, i = 1, \ldots, n.$■

Theorem 15 *Let $B = x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n}$ and $B^* = x_1^* R^{\chi_1} \oplus \ldots \oplus x_n^* R^{\chi_n}$ be mutually dual groups as it is defined above.*
For every admissible sequence of torsion elements $T = (t_1, \ldots, t_n)$ with respect to the sequence of the characteristics $\Xi = (\chi_1, \ldots, \chi_n)$ the following statements take place:

(i) *The group $B^* \oplus G_T$ is quotient divisible. Moreover, it is a direct sum of quotient divisible rank-1 subgroups. The set $x_1^* + t_1, \ldots, x_n^* + t_n$ is a basis of the group $B^* \oplus G_T$.*

(ii) *The dual to $B^* \oplus G_T$ torsion free group A_T with respect to the basis $x_1^* + t_1, \ldots, x_n^* + t_n$ is an almost completely decomposable group with the basis $x_1, \ldots, x_n$. Moreover, $B \subset A_T$ and $A_T / B \cong G_T$. Thus every admissible sequence T gives an almost completely decomposable group A_T.*

(iii) *Let A_S be an almost completely decompsable group corresponding to another admissible sequence $S = (s_1, \ldots, s_n)$ for the same sequence of the characteristics $\Xi = (\chi_1, \ldots, \chi_n)$.*
The inclusion $A_T \subset A_S$ takes place if and only if there exists a homomorphism $\eta : G_S \to G_T$ such that $\eta(s_1) = t_1, \ldots, \eta(s_n) = t_n$.

Proof The first statement of the theorem is a direct consequence of definitions and Lemma 1.

The Z-adic completion M of the group $B^* \oplus G_T$ is of the form $M = B^0 \oplus G_T$, where $B^0 = x_1^0 Z_{\chi_1} \oplus \ldots \oplus x_n^0 Z_{\chi_n}$. The set of elements $x_1^0 + t_1, \ldots, x_n^0 + t_n$ generates the module M over the ring $\widehat{Z}$ and it is a part of the triple corresponding to the quotient divisible group $B^* \oplus G_T$ with the basis $x_1^* + t_1, \ldots, x_n^* + t_n$. The set $x_1^0 + t_1, \ldots, x_n^0 + t_n$ is not necessarily linearly independent over $\widehat{Z}$, for example t_1 can be equal to t_2, but it is surely almost linearly independent over $\widehat{Z}$.

The group A_T is generated in V by all elements of the form

$$f\left(x_1^0 + t_1\right) x_1 + \ldots + f\left(x_n^0 + t_n\right) x_n, \tag{9.1}$$

where f runs through the group
$Hom_{\widehat{Z}}\left(B^0 \oplus G_T, Q/Z\right) = Hom_{\widehat{Z}}\left(B^0, Q/Z\right) \oplus Hom_{\widehat{Z}}\left(G_T, Q/Z\right)$. If the function f is running only through the first direct summand $Hom_{\widehat{Z}}\left(B^0, Q/Z\right)$, then the elements (9.1) generate in total the group $B = x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n}$. Thus the group A_T is generated by B and the finite set of elements (9.1), where f is running through $Hom_{\widehat{Z}}\left(G_T, Q/Z\right)$.

We denote $G_T^* = Hom_{\widehat{Z}}\left(G_T, Q/Z\right)$ and identify $G_T^{**} = G_T$. If $t \in G_T$ and $f \in G_T^*$, then $t : G_T^* \to Q/Z$ is defined as $t(f) = f(t)$. It is easy to see that the function $\theta : G_T^* \to V/B$, where $\theta(f) = (f(t_1) x_1 + \ldots + f(t_n) x_n) + B$, is a homomorphism. Let us prove that it is injective. Suppose $\theta(f) = 0$. It means that $f(t_1) x_1 + \ldots + f(t_n) x_n \in B$. Let $f(t_i) = \frac{k_i}{m_i} + Z$, $\gcd(k_i, m_i) = 1$. Every prime divisor p of m_i is a divisor of the order of the element t_i. Since t_i is admissible with respect to χ_i, the p-component of χ_i is zero and hence $\frac{1}{p}x_i \notin B$. This contradiction shows that $m_i = 1$ for every i and therefore $f = 0$.

Identifying along the monomorphism θ, we obtain $G_T^* \subset V/B$. The preimage of G_T^* at the natural homomorphism $V \to V/B$ is exactly the group A_T. Thus $B \subset A_T$ and $A_T/B = G_T^*$. The observation $G_T^* \cong G_T$ completes the second statement of the theorem.

Let $S = (s_1, \ldots, s_n)$ be another admissible sequence which similarly determines a group $A_S \subset V$. It is clear that $A_T \subset A_S \Leftrightarrow A_T/B \subset A_S/B$. Let $A_T \subset A_S$. Taking into account all the identifications, the embedding $id : A_T/B \to A_S/B$ can be described in the following way.

Let $f \in A_T/B = G_T^* = Hom_{\widehat{Z}}(G_T, Q/Z)$. Then $id(f)$ is such a homomorphism $g \in A_S/B = G_S^* = Hom_{\widehat{Z}}(G_S, Q/Z)$ that $(f(t_1)x_1 + \ldots + f(t_n)x_n) + B = (g(s_1)x_1 + \ldots + g(s_n)x_n) + B$, that is $(f(t_1) - g(s_1))x_1 + \ldots + (f(t_n) - g(s_n))x_n \in B$. Similarly to the injectivity of θ, it follows that $f(t_1) = g(s_1), \ldots, f(t_n) = g(s_n)$. Since $id : G_T^* \to G_S^*$ is injective, the dual homomorphism $id^* : G_S^{**} \to G_T^{**}$ is surjective. Here $id^*(h) = h \circ id$ for an element $h : G_S^* \to Q/Z$ of the group G_S^{**}. Considering $s_i \in G_S = G_S^{**}$, we have $(id^*(s_i))(f) = s_i(id(f)) = s_i(g) = g(s_i) = f(t_i) = t_i(f)$ for every $f \in G_T^*$. Thus $id^*(s_i) = t_i$ for all i and $id^* : G_S \to G_T$ is the desired homomorphism.

Conversely, if $\eta : G_S \to G_T$ is a homomorphism such that $\eta(s_1) = t_1, \ldots, \eta(s_n) = t_n$, then every generator of the form $f(t_1)x_1 + \ldots + f(t_n)x_n$ of the group A_T can be represented in the form $(f\eta)(s_1)x_1 + \ldots + (f\eta)(s_n)x_n$ as a generator of the group A_S. Therefore $A_T \subset A_S$.■

This theorem together with Theorems 2 and 11 leads to the following corollary.

Corollary 16 *Let a sequence of elements $T = (t_1, \ldots, t_n)$ be admissible with respect to a sequence of characteristics $\Xi = (\chi_1, \ldots, \chi_n)$. Then the following statements hold for the group A_T and the basis $x_1, \ldots x_n$ defined in Theorem 15*:

(i) $A_T = \langle x_1, \ldots, x_k \rangle_* \oplus \langle x_{k+1}, \ldots, x_n \rangle_*$ *if and only if* $G_T = \langle t_1, \ldots, t_k \rangle \oplus \langle t_{k+1}, \ldots, t_n \rangle$.

(ii) *The group A_T is indecomposable if the characteristics $\chi_1, \ldots, \chi_n$ belong to pairwise incomparable types and the intersections of the cyclic group $\langle t_1 \rangle$ with each of the cyclic groups $\langle t_2 \rangle, \ldots, \langle t_n \rangle$ are different from zero.*■

For every two admissible sequences $T = (t_1, \ldots, t_n)$ and $S = (s_1, \ldots, s_n)$ with respect to the same sequence of characteristics $\Xi = (\chi_1, \ldots, \chi_n)$, we define:

- $T \leq S$ if there exists a homomorphism $\eta : G_S \to G_T$ such that $\eta(s_1) = t_1, \ldots, \eta(s_n) = t_n$.
- $T \sim S$ if there exists an isomorphism $\eta : G_S \to G_T$ such that $\eta(s_1) = t_1, \ldots, \eta(s_n) = t_n$.

The second relation is an equivalence. The first relation is an order on the set of equivalence classes of admissible sequences. Thus we obtain a lattice of admissible sequences L_Ξ. We call a group of the form A_T as *admissible* with respect to Ξ.

Corollary 17 *Let $B = x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n}$ be a completely decomposable torsion free group. The lattice by inclusion of the admissible extensions of B is isomorphic to the lattice L_Ξ.*■

The restriction of admissibility is not very hard as it is shown in the following theorem.

Theorem 18 *For every almost completely decomposable group A there exist a sequence of characteristics $\Xi = (\chi_1, \ldots, \chi_n)$ and an admissible sequence of elements $T = (t_1, \ldots, t_n)$ such that $A = A_T$.*

Proof The group A contains a completely decomposable subgroup $B = x_1 R_{\kappa_1} \oplus \ldots \oplus x_n R_{\kappa_n}$ of finite index. Let $P = \{p_1, \ldots, p_m\}$ be the set of all prime divisors of this index. Replacing the finite p-components of the characteristics $\kappa_1, \ldots, \kappa_n$ by zeros for all $p \in P$, we obtain a new sequence of characteristics $\chi_1 \le \kappa_1, \ldots, \chi_n \le \kappa_n$ of the same types. Then A is a finite extension of the group $B_1 = x_1 R_{\chi_1} \oplus \ldots \oplus x_n R_{\chi_n}$ and the set of prime divisors of the index is the same $P = \{p_1, \ldots, p_m\}$. Now it is easy to see from Theorem 9 that there exists an admissible sequence T for $\Xi = (\chi_1, \ldots, \chi_n)$ such that $A = A_T$.■

10 The group of A.L.S. Corner

10.1 Partitions

We consider different prime numbers $q_1, \ldots, q_{n-k}$, where $0 < k \le n$, and a group $G = \langle t_1 \rangle \oplus \ldots \oplus \langle t_{n-k} \rangle$, where the order of t_i is q_i for $i = 1, \ldots, n-k$. The group G is cyclic itself of the order $m = q_1 \cdot \ldots \cdot q_{n-k}, G = \langle t \rangle$, where $t = t_1 + \ldots + t_{n-k}$.

Since the greatest common divisor of the integers $\frac{m}{q_1}, \frac{m}{q_2}, \ldots, \frac{m}{q_{n-k}}$ is equal to 1, there exist integers $r_1, r_2, \ldots, r_{n-k}$ such that $r_1 \frac{m}{q_1} + r_2 \frac{m}{q_2} + \ldots + r_{n-k} \frac{m}{q_{n-k}} = 1$. We are interested in the last equality just as in the sum of the integers which is equal to 1.

$$m_1 + m_2 + \ldots + m_{n-k} = 1 \tag{10.1}$$

The numbers $m_1, m_2, \ldots, m_{n-k}$ have the following property

$$m_i t_i = t_i \text{ for all } i \text{ and } m_i t_j = 0 \text{ for } i \ne j.$$

A sequence of natural numbers $(P_1, \ldots, P_{k-1})$ satisfying $0 \le P_1 \le \ldots \le P_{k-1} \le n-k$, is called a *subdivision* of the interval $[1, n-k]$ into k parts by $k-1$ *partitions* $P_1, \ldots, P_{k-1}$. We define the following four sequences for a given subdivision $(P_1, \ldots, P_{k-1})$.

(i) The sequence of non-negative integers:

$$n_1 = P_1, \ldots, n_i = P_i - P_{i-1}, \ldots, n_k = (n-k) - P_{k-1}.$$

It is easy to see that $n-k = n_1 + \ldots + n_k$ and the number of different subdivisions is $\binom{n-1}{k-1}$. It is equal to the number of different representations of $n-k$ as a sum of k non-negative integers $n-k = n_1 + \ldots + n_k$.

(ii) The sequence of integers obtained from (10.1):

$$s_1 = m_1 + \ldots + m_{P_1}, \ldots, s_i = m_{P_{i-1}+1} + \ldots + m_{P_i}, \ldots, s_k = m_{P_{k-1}+1} + \ldots + m_{n-k}.$$

Note that if $n_i = 0$ then the sum s_i is empty and we define $s_i = 0$. Obviously, $s_1 + \ldots + s_k = 1$.

(iii) The sequence of elements of the group G:

$$g_1 = t_1 + \ldots + t_{P_1}, \ldots, g_i = t_{P_{i-1}+1} + \ldots + t_{P_i}, \ldots, g_k = t_{P_{k-1}+1} + \ldots + t_{n-k}.$$

If $n_i = 0$ we define $t_i = 0$. Obviously, $g_1 + \ldots + g_k = t$.

(iv) The sequence of subsets of a linearly independent set $X = \{x_1, \ldots, x_{n-k}\}$, where $x_1, \ldots, x_{n-k}$ are vectors of a rational vector space:

$$X_1 = \{x_j \mid j \leq P_1\}, \ldots, X_i = \{x_j \mid P_{i-1} < j \leq P_i\}, \ldots, X_k = \{x_j \mid P_{k-1} < j \leq n-k\}.$$

If $n_i = 0$ then $X_i = \emptyset$.

The following properties take place for a subdivision $(P_1, \ldots, P_{k-1})$:

(i) $g_1 = s_1 t, \ldots, g_k = s_k t$,

(ii) $G = \langle g_1 \rangle \oplus \ldots \oplus \langle g_k \rangle$,

(iii) the set X_i consists of n_i elements, $X = \{x_1, \ldots, x_{n-k}\} = X_1 \cup \ldots \cup X_k$ and $X_i \cap X_j = \varnothing$ for $i \neq j$,

(iv) $\langle g_i \rangle \cap \langle t_j \rangle \neq 0 \iff x_j \in X_i$.

10.2 The Corner's group

Now we are able to define the Corner's group. Besides the set of the prime numbers $q_1, \ldots, q_{n-k}$, we fix prime numbers $p, p_1, \ldots, p_{n-k}$ such that all they are different. We also define $n - k + 1$ characteristics in the following way. The p-component of a characteristic χ is ∞ and all other components are equal to 0. The p_i-component of a characteristic χ_i is ∞ and all other components are equal to $0, i = 1, \ldots, n-k$. Then $R_\chi = Q^{(p)}, R_{\chi_i} = Q^{(p_i)}$ and $R^\chi = Q_p, R^{\chi_i} = Q_{p_i}$.

The group $B = \left(u_1 Q^{(p)} \oplus \ldots \oplus u_k Q^{(p)}\right) \oplus \left(x_1 Q^{(p_1)} \oplus \ldots \oplus x_{n-k} Q^{(p_{n-k})}\right)$ is torsion free completely decomposable with a basis $u_1, \ldots, u_k, x_1, \ldots, x_{n-k}$ of rank n, we keep here the original notation of the book [13]. The dual to B quotient divisible group is of the form $B^* = \left(u_1^* Q_p \oplus \ldots \oplus u_k^* Q_p\right) \oplus \left(x_1^* Q_{p_1} \oplus \ldots \oplus x_{n-k}^* Q_{p_{n-k}}\right)$. Thus the sequence of characteristics is $\Xi = (\chi, \ldots, \chi, \chi_1, \ldots, \chi_{n-k})$. The sequence of torsion elements $T = (t, 0, \ldots, 0, t_1, \ldots, t_{n-k})$ is admissible with respect to Ξ.

We can apply Theorem 15 and define now the group of A.L.S.Corner as $C = A_T$. That is the torsion free group dual to the group $B^* \oplus G$ with respect to the basis $u_1^* + t, u_2^*, \ldots, u_k^*, x_1^* + t_1, \ldots, x_{n-k}^* + t_{n-k}$. The basis of C dual to this one is $u_1, u_2, \ldots, u_k, x_1, \ldots, x_{n-k}$.

First of all, we can see immediately by Corollary 16 that $A_T = \langle u_1, x_1, \ldots, x_{n-k} \rangle_* \oplus \langle u_2 \rangle_* \oplus \langle u_3 \rangle_* \ldots \oplus \langle u_k \rangle_*$ is the decomposition of the Corner's group into a direct sum of k indecomposable groups, because the types $[\chi], [\chi_1], \ldots, [\chi_{n-k}]$ are pairwise incomparable.

For an arbitrary representation $n-k = n_1 + \ldots + n_k$, we change only the part $u_1, u_2, \ldots, u_k$ of the common basis $u_1, u_2, \ldots, u_k, x_1, \ldots, x_{n-k}$ of the groups B and C with help of the matrix $L = \begin{pmatrix} s_1 & s_2 & s_3 & \ldots & s_k \\ -1 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & 0 & \ldots & 1 \end{pmatrix}$ such that $\begin{pmatrix} u_1 \\ \cdots \\ u_k \end{pmatrix} = L$ $\begin{pmatrix} v_1 \\ \cdots \\ v_k \end{pmatrix}$. Thus we obtain a new common basis $v_1, \ldots, v_k, x_1, \ldots, x_{n-k}$ of the groups B and C. Since det $L = s_1 + \ldots + s_k = 1$, it follows from Corollary 6 that the dual to B and C quotient divisible groups are the same groups B^* and $C^* = B^* \oplus G$, respectively. Due to Theorem 3, the dual to $v_1, \ldots, v_k, x_1, \ldots, x_{n-k}$ bases for these two groups are $v_{1B}^*, \ldots, v_{kB}^*, x_1^*, \ldots, x_{n-k}^*$ in B^* and $v_{1C}^*, \ldots, v_{kC}^*, x_1^* + t_1, \ldots, x_{n-k}^* + t_{n-k}$ in $B^* \oplus G$, where

$$\begin{aligned} (v_{1B}^*, v_{2B}^*, \ldots, v_{kB}^*) &= (u_1^*, u_2^*, \ldots, u_k^*)\, L, \\ (v_{1C}^*, \ldots, v_{kC}^*) &= (u_1^* + t, u_2^*, \ldots, u_k^*)\, L. \end{aligned} \tag{10.2}$$

Note that by Corollary 13 we have two direct decompositions

$$B = \left(v_1 Q^{(p)} \oplus \ldots \oplus v_k Q^{(p)}\right) \oplus \left(x_1 Q^{(p_1)} \oplus \ldots \oplus x_{n-k} Q^{(p_{n-k})}\right)$$

and $B^* = (v_1^* Q_p \oplus \ldots \oplus v_k^* Q_p) \oplus \left(x_1^* Q_{p_1} \oplus \ldots \oplus x_{n-k}^* Q_{p_{n-k}}\right)$, we denote here and further $v_1^* = v_{1B}^*, \ldots, v_k^* = v_{kB}^*$. It means that we can apply Theorem 15 once more for the fixed common basis $v_1, \ldots, v_k, x_1, \ldots, x_{n-k}$ of the groups B and C. Subtracting from the second equality (10.2) the first one, we obtain $(v_{1C}^*, \ldots, v_{kC}^*) - (v_{1B}^*, v_{2B}^*, \ldots, v_{kB}^*) = (t, 0, \ldots, 0)\, L = (s_1 t, s_2 t, \ldots, s_k t) = (g_1, \ldots, g_k)$. Thus the sequence of torsion elements is $S = (g_1, \ldots, g_k, t_1, \ldots, t_{n-k})$ which is obviously admissible with respect to the same sequence of characteristics $\Xi = (\chi, \ldots, \chi, \chi_1, \ldots, \chi_{n-k})$. The group C (with the basis $v_1, \ldots, v_k, x_1, \ldots, x_{n-k}$) is dual to $B^* \oplus G$ with respect to the basis $v_1^* + g_1, v_2^* + g_2, \ldots, v_k^* + g_k, x_1^* + t_1, \ldots, x_{n-k}^* + t_{n-k}$. In other words, $C = A_S$ in terms of Theorem 15. Due to Corollary 16, the equality $G = \langle g_1 \rangle \oplus \ldots \oplus \langle g_k \rangle$ implies a direct decomposition

$$C = A_S = \langle v_1, X_1 \rangle_* \oplus \langle v_2, X_2 \rangle_* \oplus \ldots \oplus \langle v_k, X_k \rangle_* . \tag{10.3}$$

Every direct summand $\langle v_i, X_i \rangle_*$ is an extension of the group $B_i = v_i Q^{(p)} \oplus \left(\bigoplus_{x_j \in X_i} x_j Q^{(p_j)} \right)$ with help of the group $\langle g_i \rangle$. Applying Corollary 16, we can conclude that every group $\langle v_i, X_i \rangle_*$ is indecomposable, because $\langle g_i \rangle \cap \langle t_j \rangle \neq 0$ for all $x_j \in X_i$ and the types $[\chi], [\chi_1], \ldots, [\chi_{n-k}]$ are pairwise incomparable. The indecomposable summands of the decomposition (10.3) have ranks $n_1 + 1, n_2 + 1, \ldots, n_k + 1$, respectively.

Thus the Corner's group $C_{nk} = C = A_T = A_S$ depends on a pair $0 < k \leq n$ of integers, it has rank n and the following property. For every decomposition of the number $n = n_1 + \ldots + n_k$ into a sum of k positive integers, the group C_{nk} can be decomposed into a direct sum of k indecomposable subgroups of ranks $n_1, \ldots, n_k$, respectively.

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Author information

Alexander A. Fomin, Department of Mathematics, Moscow Pedagogical State University, Russia.
Email: alexander.fomin@mail.ru