T-radicals in the category of modules

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This paper investigates the properties of a special class of radicals defined in the category of modules (T-radicals). In a sense the notion of a T-radical synthesizes the definitions of an E-radical, appearing for the first time in [4], and a T-module [3].

The first section is concerned with modules over some ring S, the second one – both over the ring S and another ring R. It is suggested that R and S are associative rings with unity, modules are unitary and, unless otherwise stated, right. The category of right S-modules is designated mod-S. The word "group" means the abelian group.

1. T($\boldsymbol{F}$)-radicals

In this section modules over S are dealt with. F designates some fixed left S-module.

Definition 1.1. An S-module A is called a T(F)-*module* if $A \otimes_S F = 0$. The class of all T(F)-modules is denoted by $\mathcal{T}(F)$.

The class $\mathcal{T}(F)$ is closed under homomorphic images, extensions and direct sums. In general, it is not closed under direct products.

Let us recall some definitions of the theory of radicals.

Definition 1.2. We will say that in the category mod-S a *preradical* λ is defined, if to each S-module A is assigned its submodule $\lambda(A)$, so that for any S-homomorphism $\varphi\colon A \to B$ the relation $\varphi(\lambda(A)) \subset \lambda(B)$ holds true.

Let λ be a preradical. A class of all S-modules A for which $\lambda(A) = A$ is called λ-*radical*.

Let us consider the following (possible) properties of a preradical λ:
R1. $\lambda(\lambda(A)) = \lambda(A)$ for any $A \in$ mod-S.
R1*. $\lambda(A/\lambda(A)) = 0$ for any $A \in$ mod-S.
R2. $\lambda(B) = B \cap \lambda(A)$ for any $A \in$ mod-S and $B \subset A$.

Definition 1.3. A preradical λ is called a *radical* if R1* holds. A radical which satisfies R1 is said to be an *idempotent radical*. A preradical λ is called a *torsion* if it satisfies R2 and R1*.

It is obvious that any torsion is an idempotent radical.

$\mathrm{W}_F(A)$ will designate the sum of all submodules B of $A \in$ mod-S such that B is a $\mathrm{T}(F)$-module. W_F is an idempotent radical, and $\mathcal{T}(F)$ is its radical class [2]. From here on we will call this idempotent radical simply "T(F)-radical".

Definition 1.4. Let A be an S-module. The F*-neutralizer* of A is the set of all elements $a \in A$ such that for all $f \in F$ we have $a \otimes_S f = 0$ in the tensor product $A \otimes_S F$. This neutralizer is denoted by $\mathrm{n}_F(A)$.

The neutralizer n_F is a radical. The following equivalences are obvious:

$$\mathrm{W}_F(A) = A \iff A \in \mathcal{T}(F) \iff A \otimes_S F = 0 \iff \mathrm{n}_F(A) = A.$$

In accordance with the definition of the T(F)-radical we immediately obtain from [2] that W_F is the largest idempotent radical λ such that $\lambda(A) \subset \mathrm{n}_F(A)$ for all $A \in$ mod-S. Therefore the inclusion $\mathrm{W}_F(A) \subset \mathrm{n}_F(A)$ is always the case. The converse inclusion is not always valid.

Example 1.5. Let S be the ring of integers, $F = \mathbf{Z}(p)$ is a cyclic group of a prime order, and $A = \mathbf{Z}$ is an infinite cyclic group. We see that $\mathrm{n}_F(A) = pA$, but $\mathrm{W}_F(A) = 0$.

Proposition 1.6. *Let F be a flat module. Then*
(a) $\mathrm{n}_F(A) = \mathrm{W}_F(A)$ *for any* $A \in$ mod-S*;*
(b) W_F *is a torsion.*

In order to prove (a) and (b), it is sufficient to use the characteristic property of flat modules.

The neutralizer construction can be iterated transfinitely. Let A be a module over S. We define $\mathrm{n}^1_F(A) = \mathrm{n}_F(A)$; $\mathrm{n}^\beta_F(A) = \bigcap_{\alpha<\beta} \mathrm{n}^\alpha_F(A)$ if β is a limit ordinal; and $\mathrm{n}^\beta_F(A) = \mathrm{n}_F(\mathrm{n}^\alpha_F(A))$ if $\beta = \alpha+1$ for some ordinal number α. The descending sequence $\mathrm{n}^1_F(A), \mathrm{n}^2_F(A), \ldots, \mathrm{n}^\alpha_F(A), \ldots$ stabilizes and for some ordinal σ we have $\mathrm{n}^\sigma_F(A) = \mathrm{n}^{\sigma+1}_F(A)$. Let us introduce the notation $\mathrm{n}^\infty_F(A) = \mathrm{n}^\sigma_F(A)$.

Proposition 1.7. *For any S-module A the equality* $\mathrm{n}^\infty_F(A) = \mathrm{W}_F(A)$ *holds.*

Proof. We aim to prove the inclusion $\mathrm{W}_F(A) \subset \mathrm{n}^\alpha_F(A)$ by induction. For $\alpha = 1$ this inclusion has been proved previously. If β is a limit ordinal then, since for all $\alpha < \beta$ the required inclusion is valid, the inclusion $\mathrm{W}_F(A) \subset \mathrm{n}^\beta_F(A)$ is also valid. Assume now that for some ordinal number α we have $\beta = \alpha + 1$. The sequence of inclusions

$$\mathrm{W}_F(A) = \mathrm{W}_F(\mathrm{W}_F(A)) \subset \mathrm{W}_F(\mathrm{n}^\alpha_F(A)) \subset \mathrm{n}_F(\mathrm{n}^\alpha_F(A)) = \mathrm{n}^\beta_F(A)$$

completes the induction. For $\alpha = \sigma$ we obtain $\mathrm{W}_F(A) \subset \mathrm{n}^\sigma_F(A) = \mathrm{n}^\infty_F(A)$.

Furthermore, the following equalities hold: $\mathrm{n}_F(\mathrm{n}_F^\sigma(A)) = \mathrm{n}_F^{\sigma+1}(A) = \mathrm{n}_F^\sigma(A)$. Hence $\mathrm{n}_F^\sigma(A) = \mathrm{n}_F^\infty(A) \in \mathcal{T}(F)$, which implies that $\mathrm{n}_F^\infty(A) \subset \mathrm{W}_F(A)$. This completes the proof of the proposition. □

To conclude the section, we will consider the special case when S is the ring of integers $\mathbf{Z}$; in this situation S-modules are simply abelian groups. Let us describe all the radicals W_F. The torsion subgroup of a group A will be denoted by $\mathrm{t}(A)$, p-component of A – by $\mathrm{t}_p(A)$ or A_p. Note that all torsions λ in the category of abelian groups can be divided into two types: in the first type are those for which $\lambda(\mathbf{Z}) = 0$ (let us call such torsions *proper*), in the second type is only one torsion $\lambda(A) = A$ for all groups A. All proper torsions in the category of abelian groups have the form $\lambda(A) = \bigoplus_{p\in P} A_p$, where P is some subset (not necessarily non-empty) of the set of all primes. Conversely, any preradical λ defined in the way indicated above is a torsion in the category of abelian groups.

Proposition 1.8. *For any group F the class $\mathcal{T}(F)$ is closed under pure subgroups.*

The proposition follows immediately from the properties of purely exact sequences. Note that not all radical classes have the indicated closure property.

The following result is an immediate consequence of Proposition 1.8 and the fact that for an arbitrary idempotent radical λ the subgroup $\lambda(A)$ of a group A is always pure [1].

Proposition 1.9. *If $F = G \oplus H$, then for any group A the equalities $\mathrm{W}_F(A) = \mathrm{W}_G(\mathrm{W}_H(A)) = \mathrm{W}_H(\mathrm{W}_G(A))$ hold.*

Let F be a non-periodic abelian group. All prime numbers p can be divided into three disjoint sets:
L – those p, for which the group F is p-divisible;
M – those p, for which the group F is not p-divisible, and the factor group $F/\,\mathrm{t}(F)$ is;
N – those p, for which the factor group $F/\,\mathrm{t}(F)$ is not p-divisible.

Then for any abelian group A we have

$$\mathrm{W}_F(A) = \Big(\bigoplus_{p\in L} A_p\Big) \oplus \Big(\bigoplus_{p\in M} D_p\Big),$$

where D_p denotes the largest divisible p-subgroup of A. We will point out two important particular cases: if $F = \mathbf{Q}$ is an additive group of all rational numbers, then $\mathrm{W}_F = \mathrm{t}$; if $F = \mathbf{Q}^{(p)}$, then $\mathrm{W}_F = \mathrm{t}_p$.

Now let F be an arbitrary torsion group, $F = \bigoplus_p F_p$. It is clear that $\mathcal{T}(F) = \bigcap_p \mathcal{T}(F_p)$. The set of all prime numbers p can be divided into three disjoint sets:
L – those p, for which $F_p = 0$;
M – those p, for which F_p is a non-zero divisible group;
N – those p, for which the group F_p is not divisible.

It is easy to see that the group A is contained in the class $\mathcal{T}(F)$ if and only if the factor group $A/\operatorname{t}(A)$ is p-divisible for all $p \in M$, and the group A is p-divisible for all $p \in N$. In this case $\mathrm{W}_F(A)$ is the largest subgroup of A with these properties (it can be found as a sum of all such subgroups). Note that if F is a torsion group, then all divisible groups are $\mathrm{T}(F)$-groups.

We have obtained a complete description of $\mathrm{T}(F)$-radicals in the category of abelian groups. It is quite easy to see that this class of radicals forms in fact a set. We will see below that this set is closed under intersections.

Suppose we have a direct decomposition $F = \bigoplus_{i\in I} F_i$. In this case it is obvious that $\mathcal{T}(F) = \bigcap_{i\in I} \mathcal{T}(F_i)$. For the sake of convenience $\mathrm{T}(F_i)$-radicals will be denoted simply by W_i. The following theorem is valid.

Theorem 1.10. *For an arbitrary decomposition $F = \bigoplus_{i\in I} F_i$ and for any group A the equality $\mathrm{W}_F(A) = \bigcap_{i\in I} \mathrm{W}_i(A)$ holds.*

Let us consider when the $\mathrm{T}(F)$-radical is a torsion. It is clear that, if n_F is an idempotent radical, then for any S-module A the equality $\mathrm{n}_F^2(A) = \mathrm{n}_F(A)$ is valid, therefore $\mathrm{n}_F^\infty(A) = \mathrm{n}_F(A)$. Hence (see Proposition 1.7) the $\mathrm{T}(F)$-radical coincides with the F-neutralizer. So, if n_F is a torsion, then W_F is a torsion too. It is easy to verify that the following proposition holds.

Proposition 1.11. *The following conditions are equivalent:*
(a) $\mathrm{n}_F(A) = A$ *for any abelian group A;*
(b) $\mathrm{W}_F(A) = A$ *for any abelian group A;*
(c) $F = 0$.

Thus we know when n_F and/or W_F are "non-proper" torsions. The following theorem shows when the F-neutralizer and the $\mathrm{T}(F)$-radical are proper torsions.

Theorem 1.12. *Let S be the ring of integers, F – a group. The following conditions are equivalent:*
(a) n_F *is a proper torsion;*
(b) W_F *is a proper torsion;*
(c) the group F is non-periodic, and for any prime p, such that the factor group $F/\operatorname{t}(F)$ is p-divisible, the group F is also p-divisible.

Proposition 1.11 and Theorem 1.12 show that if S is the ring of integers, then the conditions "W_F is a torsion" and "n_F is a torsion" are equivalent. Let us make sure that in general this is not the case.

Example 1.13. Let S be the ring of residues modulo p^k (where $k > 1$), and $F = \mathbf{Z}(p)$. Then for any S-module A we have $\mathrm{W}_F(A) = 0$, hence W_F is a torsion. But $\mathrm{n}_F(A) = pA$, that is, the radical n_F is not even idempotent.

2. T-radicals

In this section we deal with two rings: S and R. Suppose that a ring homomorphism $e\colon S \to R$ is given. Any R-module A can be considered as an *attracting* S-module if we define $as = ae(s)$ for all $a \in A$, $s \in S$. It is easy to see that R and $e(S)$ are S-S-bimodules. Throughout this section it is agreed that $F = R/e(S)$.

For any module $A \in$ mod-R we can consider the canonical epimorphism $h\colon A \otimes_S R \to A \otimes_R R$.

Definition 2.1. An R-module A is called a T-*module* with respect to the ring homomorphism e if the epimorphism h is an isomorphism.

We will briefly say "T-module". A class of all T-modules over R is denoted by $\mathcal{T}$.

Definition 2.2. Let A be an R-module. The symbol $\mathrm{W}(A)$ will designate the sum of all submodules B of $A \in$ mod-R such that $B \in \mathcal{T}$. The submodule $\mathrm{W}(A)$ is called a T-*radical* of A.

The class $\mathcal{T}$ contains a module A_R if and only if $A \otimes_S F = 0$ [3]. Hence the elements of the class $\mathcal{T}$ are all those R-modules which, if treated as attracting S-modules, are contained in the class $\mathcal{T}(F)$; alternatively, $\mathcal{T} = \mathcal{T}(F) \cap$ mod-R. The latter equality and the definitions of $\mathrm{W}_F(A)$ and $\mathrm{W}(A)$ imply the inclusion $\mathrm{W}(A) \subset \mathrm{W}_F(A)$.

Theorem 2.3. *For any ordinal number α the module $\mathrm{n}_F^\alpha(A)$ is a submodule of an R-module A.*

Proof. Let us proceed by induction on α.

Base of induction ($\alpha = 1$). Symbols $\overline{r}$, $\overline{r}_1$, etc. will be used to designate elements of the left module $F = R/e(S)$. Let us fix an arbitrary element $r_1 \in R$ and define a mapping $f\colon A \times R \to A \otimes_S F$ by $f(a, r) = ar \otimes_S \overline{r}_1 - a \otimes_S \overline{rr}_1$.

The equalities

$$f(as, r) = (as)r \otimes_S \overline{r}_1 - as \otimes_S \overline{rr}_1 = asr \otimes_S \overline{r}_1 - a \otimes_S \overline{srr}_1 = f(a, sr)$$

show the mapping f to be S-balanced. Hence there exists a homomorphism of abelian groups $\varphi\colon A \otimes_S R \to A \otimes_S F$ such that $\varphi(a \otimes_S r) = ar \otimes_S \overline{r}_1 - a \otimes_S \overline{rr}_1$. The exact sequence of S-modules

$$0 \longrightarrow e(S) \xrightarrow{\alpha} R \xrightarrow{\beta} F \longrightarrow 0$$

yields the exact sequence of abelian groups

$$A \otimes_S e(S) \xrightarrow{\overline{\alpha}} A \otimes_S R \xrightarrow{\overline{\beta}} A \otimes_S F \longrightarrow 0.$$

For all $a \in A$, $s \in S$ we obtain

$$\varphi(a \otimes_S e(s)) = as \otimes_S \overline{r}_1 - a \otimes_S \overline{sr}_1 = a \otimes_S s\overline{r}_1 - a \otimes_S s\overline{r}_1 = 0.$$

Hence $\operatorname{Im} \overline{\alpha} \subset \operatorname{Ker} \varphi$, and this is equivalent to the relation $\operatorname{Ker} \overline{\beta} \subset \operatorname{Ker} \varphi$. It follows that there exists a group homomorphism $\psi\colon A \otimes_S F \to A \otimes_S F$ such that $\psi(a \otimes_S \overline{r}) = ar \otimes_S \overline{r}_1 - a \otimes_S \overline{rr}_1$. For arbitrary $b \in \mathrm{n}_F(A)$, $r \in R$ we have

$$br \otimes_S \overline{r}_1 = br \otimes_S \overline{r}_1 - b \otimes_S \overline{rr}_1 = \psi(b \otimes_S \overline{r}) = \psi(0) = 0.$$

Since the foregoing reasoning is valid for an arbitrary element $r_1 \in R$, it follows that $br \in \mathrm{n}_F(A)$. Thus $\mathrm{n}_F(A)$ is a submodule of A_R.

Induction step. If β is a limit ordinal, and for all $\alpha < \beta$ the module $\mathrm{n}_F^\alpha(A)$ is a submodule of A_R, then it is clear that this statement also holds for $\mathrm{n}_F^\beta(A)$. If $\beta = \alpha + 1$, we see that $\mathrm{n}_F^\alpha(A)$ is a submodule of A_R, and $\mathrm{n}_F^\beta(A) = \mathrm{n}_F(\mathrm{n}_F^\alpha(A))$ is a submodule of $\mathrm{n}_F^\alpha(A) \in$ mod-R. Hence $\mathrm{n}_F^\beta(A)$ is also a submodule of A_R. □

Proposition 1.7 and Theorem 2.3 produce

Corollary. *For any R-module A the equality* $\mathrm{W}_F(A) = \mathrm{W}(A)$ *is valid.*

This corollary leads us to the following result: the value of the radical $\mathrm{W}(A)$ does not depend on the way we define an R-module structure on $A \in$ mod-S (provided that this structure agrees with the existing S-module one). Moreover, we can extend the domain of the T-radical from mod-R to mod-S, identifying W_F and W.

Now it will be shown that if we have a ring S and an S-S-bimodule F, then there exist a ring R and a homomorphism $e\colon S \to R$ such that the bimodule $R/e(S)$ is isomorphic to F.

Let us define R as a set of matrices:

$$R = \left\{ \begin{pmatrix} s & f \\ 0 & s \end{pmatrix} \,\middle|\, s \in S,\ f \in F \right\}.$$

It is easy to verify that with respect to the usual operations of addition and multiplication R forms an associative ring with unity. Define a homomorphism e as follows:

$$e(s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}.$$

Then it is clear that $R/e(S) \cong F$. Note that an arbitrary S-module A can be considered as an R-module if we define

$$a \begin{pmatrix} s & f \\ 0 & s \end{pmatrix} = as$$

for all $a \in A$. Then $as = ae(s)$, which agrees with the process of defining the attracting S-module structure on R-modules.

Since modules over commutative rings can be considered as bimodules, we obtain the following result: if S is a commutative ring, then any $\mathrm{T}(F)$-radical has the form of a T-radical (if the latter is treated as the one defined in the category of S-modules) for some ring R and some embedding $e\colon S \to R$.

References

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