Isomorphism of a Ring to the Endomorphism Ring of an Abelian Group

V. M. Misyakov

UDK 512.541+512.541.52+512.552.13+512.552.16

Key words: associative ring, endomorphism ring, abelian group.

Аннотация

This paper presents necessary and sufficient conditions under which isomorphism of endomorphism rings of additive groups of arbitrary associative rings with 1 implies isomorphism of these rings. For a certain class of abelian groups, we present a criterion which shows when isomorphism of their endomorphism rings implies isomorphism of these groups. We demonstrate necessary and sufficient conditions under which an arbitrary ring is the endomorphism ring of an abelian group. This solves Problem 84 in [4].

All the rings in this paper are associative with 1. If R is a ring (abelian group), then $E(R)$ is the endomorphism ring of its additive group, R^+ is the additive group of the ring R, and R^l is the subring of left multiplications of the ring $E(R)$.

According to L. Fuchs [4], if two abelian groups are isomorphic, their endomorphism rings are also isomorphic. The inverse is not always true. In particular, it follows that isomorphism of rings implies isomorphism of endomorphism rings of their additive groups. So there arises a problem to obtain necessary and sufficient conditions under which isomorphism of the endomorphism rings of additive groups of the rings implies isomorphism of these rings. This problem is solved in Theorem 2. Note that if R is a ring with 1, the ring R is isomorphic to the ring of left multiplications R^l (Lemma 3.7.3 in [1]).

Lemma 1. Let in the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\alpha} & R^l & \xrightarrow{i} & E(R) \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\delta} & & \big\downarrow{\scriptstyle\beta^*} \\ S & \xrightarrow{\gamma} & S^l & \xrightarrow{i'} & E(S) \end{array} \qquad (*)$$

α, γ be ring endomorphisms; i, $i^{'}$ be identical immersions of rings; then the following are valid:
1) if β is an isomorphism of the rings R and S, then there exist isomorphisms δ and β^* such that the diagram $(*)$ becomes commutative;
2) if δ is an isomorphism of the rings R^l and S^l, then there exist isomorphisms β and β^* such that the diagram $(*)$ becomes commutative.

Proof. 1) Let $\beta : R \to S$ be a ring isomorphism, then the isomorphisms α, β, γ induce an isomorphism $\delta : R^l \to S^l$ such that $\delta = \gamma\beta\alpha^{-1}$. Then for each $r \in R$ we have: $\delta(\alpha(r)) = \delta(r^l) = \gamma(\beta(\alpha^{-1}(r^l))) = \gamma(\beta(r)) = \gamma(s) = s^l$ and, on the other hand, $\gamma(\beta(r)) = \gamma(s) = s^l$, i.e., $\delta\alpha = \gamma\beta$. The isomorphism of the rings R and S implies isomorphism of their additive groups. The latter, in its turn, implies the isomorphism $\beta^* : E(R) \to E(S)$ such that $\beta^*(\psi) = \beta\psi\beta^{-1}$ for all $\psi \in E(R)$ [4]. Let $r^l \in R^l$ and $\delta(r^l) = s^l$, then $\gamma(\beta(\alpha^{-1}(r^l))) = s^l$, $\beta(\alpha^{-1}(r^l)) = \gamma^{-1}(s^l)$, therefore, $\beta(r) = s$. Let us demonstrate that $\beta^*|_{R^l} = \delta$, i.e., $\beta^*(r^l) = s^l$. Let $x \in S$, then $(\beta^*(r^l))(x) = \beta(r^l(\beta^{-1}(x))) = \beta(r^l(y)) = \beta(ry) = \beta(r)\beta(y) = sx = s^l(x)$, i.e., $\beta^*(r^l) = s^l$. The equality $\beta^*|_{R^l} = \delta$ demonstrates that the right square of the diagram $(*)$ is commutative, then the whole diagram $(*)$ is commutative.
2) Let $\delta : R^l \to S^l$ be a ring isomorphism, then the isomorphisms α, δ, γ induce an isomorphism $\beta : R \to S$ such that $\beta = \gamma^{-1}\delta\alpha$, what makes the left square of the diagram $(*)$ commutative. Indeed, for each $r \in R$ we have: $\delta(\alpha(r)) = \delta(r^l) = s^l$ and, on the other hand, $\gamma(\beta(r)) = \gamma(\gamma^{-1}(\delta(\alpha(r)))) = \delta(\alpha(r)) = s^l$, i.e., $\delta\alpha = \gamma\beta$. Then there exists an isomorphism $\beta^* : E(R) \to E(S)$ such that $\beta^*(\psi) = \beta\psi\beta^{-1}$ for all $\psi \in E(R)$. Let $r^l \in R^l$ and $\delta(r^l) = s^l$, then $\beta(r) = \gamma^{-1}(\delta(\alpha(r))) = \gamma^{-1}(\delta(r^l)) = \gamma^{-1}(s^l) = s$, i.e., $\beta(r) = s$. Let us prove that $\beta^*|_{R^l} = \delta$, i.e., $\beta^*(r^l) = s^l$. Let $x \in S$, then $(\beta^*(r^l))(x) = (\beta r^l \beta^{-1})(x) = ((\gamma^{-1}\delta\alpha)r^l(\gamma^{-1}\delta\alpha)^{-1})(x) = (\gamma^{-1}\delta\alpha r^l\alpha^{-1}\delta^{-1}\gamma)(x) = (\gamma^{-1}\delta\alpha r^l\alpha^{-1}\delta^{-1})(x^l) = (\gamma^{-1}\delta\alpha r^l\alpha^{-1})(y^l) = (\gamma^{-1}\delta\alpha r^l)(y) = (\gamma^{-1}\delta\alpha)(ry) = (\gamma^{-1}\delta\alpha)(r)(\gamma^{-1}\delta\alpha)(y) = (\gamma^{-1}\delta)(r^l)(\gamma^{-1}\delta)(y^l) = (\gamma^{-1})(s^l)(\gamma^{-1})(x^l) = sx = s^l(x)$. Thus, the right square in the diagram $(*)$ is commutative, so the whole diagram is also commutative.

Theorem 2. For the rings R and S, the following conditions are equivalent:
1) $R \cong S$;
2) $R^l \cong S^l$;
3) $E(R) \overset{\beta}{\cong} E(S), E(R)/R^l \overset{\gamma}{\cong} E(S)/S^l$; here $\gamma\pi = \pi^{'}\beta$, where $\pi : E(R) \to E(R)/R^l$, $\pi^{'} : E(S) \to E(S)/S^l$ are canonical epimorphisms and γ is a group

isomorphism.

Proof. Equivalence of 1) and 2) follows from Lemma 1. We prove that 2)$\Rightarrow$ 3). Let $\alpha : R^l \to S^l$ be an ring endomorphism; then, by Lemma 1, there exists an isomorphism $\beta : E(R) \to E(S)$ such that the diagram with exact strings

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^l & \xrightarrow{i} & E(R) & \xrightarrow{\pi} & E(R)/R^l & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & S^l & \xrightarrow{i^{'}} & E(S) & \xrightarrow{\pi^{'}} & E(S)/S^l & \longrightarrow & 0
\end{array} \qquad (**)$$

is commutative, where γ is not given; $i, i^{'}$ are identical ring immersions; $\pi, \pi^{'}$ are canonical epimorphisms of the groups. Then, by Proposition 3 [3], there exists a group isomorphism γ such that the right equare of the diagram $(**)$ is commutative, i.e., $\gamma\pi = \pi^{'}\beta$.
3)$\Rightarrow$ 2). Let's consider the commutative diagram $(**)$ with exact strings where α is not given; $i, i^{'}$ are identical ring immersions; $\pi, \pi^{'}$ are canonical epimorphisms of the groups. Then, by Proposition 2 [3], there exists a group monomorphism $\alpha : R^l \to S^l$ such that $i^{'}\alpha = \beta i$, i.e., the diagram $(**)$ is commutative. Since $im\ \beta i \subseteq im\ i^{'}$, $\alpha = (i^{'})^{-1}\beta i$. Since $(i^{'})^{-1}, \beta, i$ are ring homomorphisms, α is a ring homomorphism. Let us prove that α is an epimorphism. Since β is an epimorphism, for an arbitrary $s \in S^l$ there exists $b \in E(R)$ such that $i^{'}(s) = \beta(b)$. Then $0 = \pi^{'}(i^{'}(s)) = \pi^{'}(\beta(b)) = \gamma(\pi(b))$ and $\pi(b) \in Ker\ \gamma$. Since γ is a monomorphism, $\pi(b) = 0$ and $b \in Ker\ \pi = im\ i$. Therefore, there exists $r \in R^l$ such that $i(r) = b$. Then $\beta(i(r)) = \beta(b) = i^{'}(s)$ and, so, $(i^{'})^{-1}(\beta(i(r))) = s$, i.e., $\alpha(r) = s$. Thus we prove that α is a ring isomorphism.

The result below refers to the so-called isomorphism theorem for endomorphismrings. By this theorem one usually means that two groups (maybe from a given class) are isomorphic if their endomorphism rings are isomorphic [2]. Let K be the class of abelian groups permitting the structure of a ring with 1. Then, for groups from this class, the following statement is valid.

Corollary 3. For all $A, B \in K$, the following are equivalent:
1) $A \cong B$;
2) $(A^l)^+ \cong (B^l)^+$;

3) $E(A) \overset{\beta}{\cong} E(B)$, $E(A)/(A^l)^+ \overset{\gamma}{\cong} E(B)/(B^l)^+$, $\gamma\pi = \pi^{'}\beta$, where $\pi : E(A) \to E(A)/(A^l)^+$, $\pi^{'} : E(B) \to E(B)/(B^l)^+$ are canonical epimorphisms.

L. Fuchs put Problem 84 [4]: find criteria for different types of rings under which these rings are endomorphism rings of abelian groups. This problem is solved in the following statement for arbitrary associative rings with 1.

Corollary 4. For a ring R and abelian group A the following are equivalent:
1) $R \cong E(A)$;
2) $R^l \cong (E(A))^l$;
3) $E(R) \overset{\beta}{\cong} E(E(A))$, $E(R)/R^l \overset{\gamma}{\cong} E(E(A))/(E(A))^l$, $\gamma\pi = \pi^{'}\beta$, where $\pi : E(R) \to E(R)/R^l$, $\pi^{'} : E(E(A)) \to E(E(A))/(E(A))^l$ are canonical epimorphisms.

If we take an associative ring A with 1 instead of an abelian group in Corollary 4, we obtain the conditions under which a ring R is the endomorphism ring of the additive group of a ring A.

References

1. F. Kasch, Modules and Rings, Mir, Moscow, 368 (1981).
2. P. A. Krylov, A. V. Mikhalev, and A. A. Tuganbaev, Connections of Abelian Groups and Their Endomorphism Rings, State University, Tomsk, 464 (2002).
3. L. A. Skornyakov, "Lectures on Homological Algebra,"Mat. vestnik, 5, №1, 71-113 (1968).
4. L. Fuchs, Infinite Abelian Groups, vol. 2, Mir, Moscow, 416 (1977).