On regularity of the center of the endomorphism ring of an abelian group

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Abstract

We study abelian groups with a regular center of the endomorphism ring.

This paper deals with a question posed in Problem 16 [1]: "For which groups centers of their endomorphism rings are regular?" We recall that a ring R is said to be regular if for each element $x \in R$ there exists an element $y \in R$ such that $xyx = x$. Since it is well-known that the center of a regular ring is regular, the class of abelian groups with a regular center of the endomorphism ring contains the class of abelian groups with a regular endomorphism ring. At the same time, as it is shown below, these classes coincide in some cases.

All the groups throughout the paper are abelian. We use the following designations: $\mathbb{Q}$ is the field of rational numbers; direct sum and product of groups (rings) are denoted by the symbols $\oplus$ and $\times$ or $\prod$ respectively. Let X and Y be two groups. Then $T(X)$ is the torsion subgroup of the group X, $T_p(X)$ is the p-component of $T(X)$, $\mathrm{E}(X)$ is the endomorphism ring of the group X, $\mathrm{Hom}(X, Y)$ is the group of homomorphisms from X to Y, $X[p] = \{a \in X \mid pa = 0\}$, $\mathrm{C}(R)$ is the center of the ring R. All not defined notions ("reduced group", "height matrix of an element" etc.) can be found in [5], [6], [7].

Lemma 1. *The center of the endomorphism ring of a group G is regular if and only if $G = \mathrm{im}(\alpha) \oplus \ker(\alpha)$ for each $\alpha \in \mathrm{C}(\mathrm{E}(G))$.*

Proof. Let $\mathrm{C}(\mathrm{E}(G))$ be a regular ring. Then for each $\alpha \in \mathrm{C}(\mathrm{E}(G))$ there exists $\beta \in \mathrm{C}(\mathrm{E}(G))$ such that $\alpha\beta\alpha = \alpha$. The inclusions $\mathrm{im}(\alpha) = \mathrm{im}(\alpha\beta\alpha) \subseteq \mathrm{im}(\alpha\beta) \subseteq \mathrm{im}(\alpha)$ and $\ker(\alpha) \subseteq \ker(\beta\alpha) \subseteq \ker(\alpha\beta\alpha) = \ker(\alpha)$ are valid, so $\mathrm{im}(\alpha) = \mathrm{im}(\alpha\beta)$ and $\ker(\alpha) = \ker(\beta\alpha) = \ker(\alpha\beta)$. Since $\alpha\beta$ is an idempotent of the ring $\mathrm{C}(\mathrm{E}(G))$, then $G = \mathrm{im}(\alpha\beta) \oplus \ker(\alpha\beta) = \mathrm{im}(\alpha) \oplus \ker(\alpha)$.

Conversely, let $G = \mathrm{im}(\alpha) \oplus \ker(\alpha)$ for each $0 \neq \alpha \in \mathrm{C}(\mathrm{E}(G))$. Then $\alpha|_{\mathrm{im}(\alpha)}$ is an automorphism on $\mathrm{im}(\alpha)$. Therefore, there exists $\beta \in \mathrm{E}(G)$ which annihilates $\ker(\alpha)$ and is inverse to $\alpha|_{\mathrm{im}(\alpha)}$ on $\mathrm{im}(\alpha)$. We demonstrate that $\beta \in \mathrm{C}(\mathrm{E}(G))$. Let $\varphi \in \mathrm{E}(G)$ and $x \in G$, then $x = x_1 + x_2$, where $x_1 \in \mathrm{im}(\alpha)$, $x_2 \in \ker(\alpha)$. Therefore, $(\varphi\beta)(x) = \varphi(\beta(x_1 + x_2)) = \varphi(\beta(x_1))$. Since $x_1 \in \mathrm{im}(\alpha)$, there exists $a_1 \in \mathrm{im}(\alpha)$ such that $\alpha|_{\mathrm{im}(\alpha)}(a_1) = x_1$. Since $\mathrm{im}(\alpha)$, $\ker(\alpha)$ are fully invariant subgroups in G, $\varphi(\beta(x_1)) = \varphi(\beta(\alpha|_{\mathrm{im}(\alpha)}(a_1))) = \varphi(a_1) = (\beta\alpha)(\varphi(a_1)) = \beta(\varphi(\alpha(a_1))) = \beta(\varphi(x_1)) = \beta(\varphi(x_1)) + \beta(\varphi(x_2)) = (\beta\varphi)(x)$.

Let $a \in G$, then $a = a_1 + a_2$, where $a_1 \in \mathrm{im}(\alpha)$, $a_2 \in \ker(\alpha)$. Therefore, $(\alpha\beta\alpha)(a) = (\alpha\beta\alpha)(a_1) = \alpha(a_1) = \alpha(a_1) + \alpha(a_2) = \alpha(a)$. If $\alpha = 0$, the statement is trivial.

Note 1. As it was shown in [2] and [3], the endomorphism ring of an elementary or divisible torsion free group is regular. So, the center of this ring is also regular.

Note 2. We shall use the following statement which is easy to prove. Let $G = A \oplus B$, where A, B are fully invariant subgroups of the group G. The ring $\mathrm{C}(\mathrm{E}(G))$ is regular if and only if $\mathrm{C}(\mathrm{E}(A))$, $\mathrm{C}(\mathrm{E}(B))$ are regular rings.

The following statement is based on the concept of idealization of bimodules. Let us recall this concept.

Let R and S be two rings, M be an R-S-bimodule. Idealization of the bimodule M is a ring consisting of all matrices of the form $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$, where $r \in R$, $s \in S$, $m \in M$, with usual operations of matrix addition and multiplication. This ring will be denoted by the symbol $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ or by a single letter K. The rings R and S will be naturally identified with $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$

respectively, the product $R \times S$ with $\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$, and the bimodule M with $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. To save place, the diagonal matrix $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$ will be written in the vector form $(r,\, s)$. There are two canonical surjective ring homomorphisms:

$$K \to R,\ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \to r,\ K \to S,\ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \to s.$$

The following Lemma was proved in [8].

Lemma 2. *The center* $\mathrm{C}\,(K)$ *of the idealization* K *of the* R-S*-bimodule* M *consists of all diagonal matrices* $(r,\, s)$, *where* $r \in \mathrm{C}\,(R)$, $s \in \mathrm{C}\,(S)$, *and* $rm = ms$ *for each* $m \in M$.

If A is a left R-module, its annihilator is denoted by $\mathrm{Ann}_{\,R}\, A$. As follows from the Lemma, $\mathrm{C}\,(K)$ is a subring in $\mathrm{C}\,(R) \times \mathrm{C}\,(S)$ and, similarly, $\mathrm{Ann}_{\,\mathrm{C}\,(R)}\, M$, $\mathrm{Ann}_{\,\mathrm{C}\,(S)}\, M \subseteq \mathrm{C}\,(K)$. Taking into account the note before Lemma 2, we have the ring homomorphisms $f : \mathrm{C}\,(K) \to \mathrm{C}\,(R)$, $(r,\, s) \to r$ and $g : \mathrm{C}\,(K) \to \mathrm{C}\,(S)$, $(r,\, s) \to s$. The annihilators $\mathrm{Ann}_{\,\mathrm{C}\,(R)}\, M$ and $\mathrm{Ann}_{\,\mathrm{C}\,(S)}\, M$ are fixed under the homomorphisms f and g respectively. In particular, $\mathrm{Ann}_{\,\mathrm{C}\,(R)}\, M \subseteq \mathrm{im}\,(f)$, $\mathrm{Ann}_{\,\mathrm{C}\,(S)}\, M \subseteq \mathrm{im}\,(g)$. We present one more result from [8], which will be referred to below.

Lemma 3. *In the above-stated designations, we have*
1) $\ker\,(f) = \mathrm{Ann}_{\,\mathrm{C}\,(S)}\, M$ *and* $\ker\,(g) = \mathrm{Ann}_{\,\mathrm{C}\,(R)}\, M$;
2) if M *is an exact* $\mathrm{C}\,(S)$*-module, then* f *is a monomorphism; and if* M *is an exact* $\mathrm{C}\,(R)$*-module, then* g *is a monomorphism.*

Considering endomorphism rings of groups, we can obtain an idealization of a bimodule in the following situation. Let G be a direct sum of two groups, $G = B \oplus A$, where B is a fully invariant summand, i.e., $\mathrm{Hom}\,(B,\, A) = 0$. The homomorphism group $\mathrm{Hom}\,(A,\, B)$ can be turned to an $\mathrm{E}\,(B)$-$\mathrm{E}\,(A)$-bimodule in the standard way. Therefore, one can write an idealization of this bimodule:

$$\begin{pmatrix} \mathrm{E}\,(B) & \mathrm{Hom}\,(A,\, B) \\ 0 & \mathrm{E}\,(A) \end{pmatrix}.$$

Since it is well-known that the ring $\mathrm{E}(G)$ can be naturally identified with a given ring of matrices (see [6, Theorem 106.1]), the endomorphism ring $\mathrm{E}(G)$ can be considered as an idealization of the $\mathrm{E}(B)$-$\mathrm{E}(A)$-bimodule $\mathrm{Hom}(A, B)$.

Proposition 4. *Let $G = B \oplus A$, where A is a reduced torsion free group, B is a divisible torsion free group. Then the center of the endomorphism ring of the group G can be identified with a subring of the field $\mathbb{Q}$ generated by 1 and all numbers $1/p$ such that $p\,G = G$.*

Proof. Let $G = B \oplus A$, where A and B satisfy the condition of Proposition 8 and $B \neq 0$. Since $\mathrm{Hom}(B, A) = 0$, we conclude that the ring $\mathrm{E}(G)$ is an idealization of the $\mathrm{E}(B)$-$\mathrm{E}(A)$-bimodule $\mathrm{Hom}(A, B)$. Let us demonstrate that $\mathrm{Hom}(A, B)$ is an exact $\mathrm{E}(A)$-module. Suppose the contrary, i.e., $\mathrm{Hom}(A, B)\,\alpha = 0$ for some $0 \neq \alpha \in \mathrm{E}(A)$. It is easy to show that there exists an element $a \in A$ such that $\circ(a) = \infty$ and $\alpha(a) \neq 0$. Let $0 \neq b \in B$. Since the height matrix of the element $\alpha(a)$ in the group A is less than that of the element b in the group B, there exists $\varphi \in \mathrm{Hom}(A, B)$ such that $\varphi(\alpha(a)) = b$. Since $b \neq 0$, then $\varphi\alpha \neq 0$, what contradicts the assumption. Thus, $\mathrm{Hom}(A, B)$ is an $\mathrm{E}(A)$-module. By Lemma 3, the ring $\mathrm{C}(\mathrm{E}(G))$ can be embedded into the ring $\mathrm{C}(\mathrm{E}(B))$ isomorphic to $\mathbb{Q}$. Let us show that the ring $\mathrm{C}(\mathrm{E}(G))$ can be identified with a subring of the field $\mathbb{Q}$. Since all the natural numbers belong to $\mathrm{C}(\mathrm{E}(G))$, it is sufficient to show that if a rational number of the form $\frac{1}{q} \in \mathrm{im}(\varphi)$, where φ is an embedding of the ring $\mathrm{C}(\mathrm{E}(G))$ into the field $\mathbb{Q}$, there exists an endomorphism from $\mathrm{C}(\mathrm{E}(G))$ which acts on elements of the group G as multiplication by the fraction $\frac{1}{q}$. Let the equation $qx = 1$ (where q is a prime number) be solved in $\mathrm{im}(\varphi)$, i.e., there exists $\beta \in \mathrm{im}(\varphi)$ such that $q\beta = 1$. Since $\varphi(\alpha) = \beta$ for some $\alpha \in \mathrm{C}(\mathrm{E}(G))$, it means that α is a solution of the equation $qy = 1$ in the ring $\mathrm{C}(\mathrm{E}(G))$. Then for each $g \in G$, $\alpha(g) = (\frac{1}{q}\,q)(\alpha(g)) = \frac{1}{q}(q\alpha)(g) = \frac{1}{q}(g)$. Therefore, elements of the center represent multiplications by rational numbers m/n. Moreover, it is easy to see that $nG = G$. Conversely, if $nG = G$, then the result of multiplication of the group G by a number m/n is in $\mathrm{C}(\mathrm{E}(G))$. We obtain that the center $\mathrm{C}(\mathrm{E}(G))$ can be identified with a subring of the field $\mathbb{Q}$ mentioned in the Proposition.

Now we prove the main result of the work.

Theorem 5. *1) If G is not a reduced group, then* $\mathrm{C}\,(\mathrm{E}\,(G))$ *is a regular ring if and only if the group G satisfies at least one of the following conditions:*
a) G is a divisible torsion free group;
b) $G = A \oplus D$, where A is an elementary group, and D is a divisible torsion free group.
2) If G is a reduced group and $\mathrm{C}\,(\mathrm{E}\,(G))$ *is a regular ring, then $T\,(G)$ is an elementary group, $G\,/\,T\,(G)$ is a divisible group and* $\bigoplus_{p\in P} T_p\,(G) \subseteq G \subseteq \prod_{p\in P} T_p\,(G)$.

Proof. Note that the proof of this Theorem is based on the idea of the corresponding proofs from [2] and [3]. For instance, replacing E (G) by its center, we can obtain the proof of the condition 2) of the Theorem. For completeness, we prove condition 1). Let $G = A \oplus D$, where A is a reduced group, $0 \neq D$ is a divisible group. Assume that $T\,(G) = 0$. Let n be an arbitrary natural number. Since $n \in \mathrm{C}\,(\mathrm{E}\,(G))$, regularity of $\mathrm{C}\,(\mathrm{E}\,(G))$ implies that $G = \mathrm{im}\,(n) \oplus \ker\,(n)$. By assumption, G is a torsion free group, so $\ker\,(n) = 0$ and $G = \mathrm{im}\,(n) = nG$, i.e., G is a divisible torsion free group. Let $T\,(G) \neq 0$. Then multiplication by a fixed prime number p is an endomorphism from $\mathrm{C}\,(\mathrm{E}\,(G))$ whose kernel is $T_p\,(G)\,[p]$. It follows from regularity of the ring $\mathrm{C}\,(\mathrm{E}\,(G))$ that $T_p\,(G)\,[p]$ is a direct summand but it is possible only if $T_p\,(G)$ coincides with $T_p\,(G)\,[p]$. Therefore, $T\,(G)$ is an elementary group. Since elementary groups are not divisible, D is a divisible torsion free group and $T\,(G) = T\,(A)$, $A \neq 0$. Let us assume that $A \neq T\,(A)$. Then, by Proposition 4, the ring $\mathrm{C}\,(\mathrm{E}\,(G))$ is isomorphic to a subring L of the field $\mathbb{Q}$ which is generated by 1 and all numbers $1/p$ such that $p\,G = G$. If we assume that $L \neq \mathbb{Q}$, there exists a prime number q such that $1/q \notin L$. Regularity of the ring L implies existence of $x \in L$ such that $qxq = q$. It follows that $x = 1/q \in L$ what contradicts the assumption. Thus, $\mathrm{C}\,(\mathrm{E}\,(G)) \cong \mathbb{Q}$ and $p\,G = G$ for any prime number p, but this is impossible because the direct summand A is reduced in G. Therefore, $A = T\,(A)$ is an elementary group. The converse statement follows from Notes 1 and 2.

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